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DESIGN AND ESTIMATION IN SMALL SAMPLE QUANTAL RESPONSE PROBLEMS: A MONTE CARLO STUDY

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In sensitivity testing for the Department of Defense, the high cost of experimental units necessitates the use of small sample sizes and accentuates the importance of design. This paper compares five data collection--estimation procedures. Four of these are modifications of the Robbins-Monro method, and the other is the Langlie. The simulation study is designed as a factorial experiment with response function, sample size, initial design point, gate width, and noise as factors. The estimated V_{50} and its root mean square error are the responses compared to assess the small sample behavior of each method. Although there is no single clear-cut winner, the Delayed Robbins-Monro ~~(DRM)~~ with maximum likelihood estimation and the Estimated Quantal Response Curve (Wu 1985) are shown to perform well over a broad variety of conditions. Keywords:

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I. INTRODUCTION

In the language of Dixon and Mood (1948) a sensitivity testing environment may be described as follows. The sensitivity of explosives to shock is tested by dropping a weight on the explosive from various heights above it. The result is either a "response" (explosion) or a "nonresponse". It is assumed that there exists for each explosive some critical height, above which the explosive will always detonate. Critical height is then a continuous variable which cannot be measured directly but rather only through observing response or nonresponse for the various levels of height. Commonly assumed is that a continuous monotone increasing function F(x) represents the probability of observing a response for each height x. The ultimate goal of sensitivity testing is to adequately describe some interval of this function. In this manner, the sensitivity of the explosive to levels of shock is modeled.

Since the Up and Down Method (1948) appeared in the literature, several other data collection and estimation procedures have been proposed including Robbins-Monro (1951), Langlie (1962), Wetherill (1963), and Wu (1985). Many of these procedures have been subjected to extensive study and comparison. However, much of the work is concerned with asymptotic properties, for example, Chung (1954) and Hodges and Lehmann (1955) concerning the Stochastic Approximation Method of Robbins and Monro. Unfortunately, Langlie's procedure, the Army standard, has been ignored to a great extent in the literature. It is the purpose of this study to compare the performance of these and more recent procedures under the constraints imposed in a field test environment.

In consideration of the Army test environment, a more practical accounting of test condition effects must be made than is given in many of the previous studies. Because of the cost of the munitions used for testing, only small samples are possible, and hence asymptotic results add little to performance understanding. In some studies maximum likelihood estimates are compared to others, but only when the maximum likelihood estimate exists uniquely. Evaluation of techniques in our expensive test environment must consider back-up estimation procedures when the necessary conditions for the intended estimator are not satisfied. Also for penetrator-against-plate testing, a specified stress (velocity of the penetrator) cannot be guaranteed. For instance, one may intend to fire a round at 2500 fps and then assess his results (penetration or nonpenetration) so that a next stress level (Next Stress) can be chosen. However, since the velocity of a round cannot be guaranteed by varying the propellant charge weight, there is white noise associated with the actual value of the stress level considered. It is important to examine how sensitive the results are to this uncontrollable white noise. The feasibility of maximum likelihood estimation under these conditions is discussed by Golub and Grubbs (1956).

The overriding consideration in preparation of this study was to create a representative slice of real-life situations wherein data collection and estimation procedures could be compared. Several interesting small sample studies were drawn upon for this purpose. Dixon (1965) and Hampton (1967) look at the effect of misplacement error (poor choice in starting value) with regard to the sequential Up and

Down Method. Dixon also looks at the effect of a poor guesstimate of the response distribution standard deviation; this controls the size of change in stress made for the next design point. Davis (1969) considers sensitivity to response distribution assumptions in addition to the above two errors for several procedures proposed at that time. Sample sizes considered in all three were relatively small. We will use these devices along with some additional ones. Additional devices include the use of white noise in the stress levels and the use of backup estimation procedures when the intended estimation fails. Also, some specific sample sizes are considered. These are sample sizes recommended recently by standard operating procedures. This, intended as a practical study, considers actual data as well as Monte Carlo simulations. All designs and estimation procedures will be applied to an empirical distribution formed from testing performed at Aberdeen Proving Ground (APG). Finally, the standard test strategy, Langlie's One Shot Test Strategy, will be compared to more recent designs proposed in the literature such as the Delayed Robbins-Monro of Cochran and Davis (1964) and the Estimated Quantal Response Curve of Wu (1985). With these considerations in mind procedures are compared according to their ability to estimate the ballistic limit V_{50}, the velocity at which half of the projectiles would be expected to penetrate the armor, i.e., the median of the response distribution F(x).

Because of the number of factors which comprise the real-life scenario, it is appropriate to introduce all of the factor levels in a diagram. See Figure 1. This is intended as an aid to the readers so that they can more easily keep the sections that follow in proper perspective. In those sections, elements of the diagram are discussed in more detail.

II. DATA COLLECTION AND ESTIMATION PROCEDURES

Data collection and estimation should be considered separate processes. It is true in the case of some test strategies that the next design point chosen is also the estimate of the target quantile. This of course need not be the case. With each data collection technique, three estimation procedures are employed: NMLE, AVR, Next Stress. NMLE is the maximum likelihood estimator of V_{50} when a normal distribution is assumed for the distribution of critical velocity. AVR is an averaging of stress levels thought to be near the median. Next Stress will vary depending on which data collection procedure is being employed. For instance in the case of the Langlie procedure it is the next design point of an averaging algorithm, whereas for Wu's technique it is intended as the logit-based maximum likelihood estimate computed using the information gathered up until that point.

Five designs will be examined in this study: Langlie, Delayed Robbins-Monro (DRM), Adaptive Robbins-Monro (ARM), Estimated Quantal Response Curve-Delayed Robbins-Monro (EQRC-DRM), and Estimated Quantal Response Curve-Adaptive Robbins-Monro (EQRC-ARM). The first is the method currently used by the U.S. Department of the Army in penetrator-against-plate testing. The second is a slight adjustment of a well known and widely used technique. The final three are fairly recent attempts

Design

(levels)	DRM	ARM	Langlie	EQRC-DRM	EQRC-ARM

Estimation

(levels)	NMLE	AVR	Next Stress

Test Condition Factors

	Sample Size	Noise	Starting Value (SV)	Gate Width (GW)	Response Function
(levels)	9	zero	V_{50}	2σ	normal
	12	normal	$V_{50}-1\sigma$	6σ	uniform
	15	asymmetric	$V_{50}-3\sigma$	10σ	Cauchy
			$V_{50}+3\sigma$		exponential
					empirical

Figure 1. Design and estimation over various test conditions

at improving on the Robbins-Monro (RM) procedure. Details for each design interfaced with the three estimation procedures follow.

1. Langlie's One Shot Test Strategy

Langlie's One Shot Test Strategy was introduced in 1962. The experimenter is asked to make a guesstimate of the location of the median of the response distribution. In practice, the actual questions are "At what stress can we be reasonably certain that a response will always occur?" and "At what stress can we be reasonably certain that a response will never occur?". The two stresses are referred to as the upper and lower gates, respectively. All design points using the Langlie strategy must reside within these two gates. The first design point, the experimenter's estimate of the median, is taken to be the average of the upper and lower gates. Subsequent points x_{n+1} are taken to be the average of x_n and x_{n-i}, where i is the first time in searching back in the data that the number of responses equals the number of nonresponses between, inclusively, x_n and x_{n-i}. If no such i exists, x_n is averaged with the upper gate for x_n, nonresponse, and the lower gate for x_n, response. The tendency of experimenters in the past has been to pick extremely wide gates, equivalent to $V_{50} \pm 5\sigma$ or more, because of the gate restriction on possible design points.

The intended estimation for the Langlie method is maximum likelihood with an assumed normal response distribution for the critical velocity. For a response function P with mean μ standard deviation σ and stress levels x_i, the likelihood function discussed by Dixon and Mood (1948) and Golub and Grubbs (1956) is given by

$$P = \prod_i p_i^{d_i} q_i^{1-d_i} \text{ with } d_i = (0, 1) = (\text{nonresponse, response}) \tag{1}$$

$$\text{and } p_i = \int_{-\infty}^{t_i} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = 1-q^i, \text{ where } t_i = \frac{x_i - \mu}{\sigma}. \tag{2}$$

The likelihood equations formed are solved iteratively using the Newton-Raphson procedure. There are several algorithms available which perform this procedure. A problem with many is slow convergence or failure to converge in many practical situations. The algorithm of DiDonato and Jarnagin (1972) as prepared for local computing facilities by McKaig and Thomas (1983) was used in this study. The claimed advantage is guaranteed global convergence to "best" estimates when they exist uniquely, even for poor initial starting values. This claim was based on extensive testing performed by the original authors. Unique estimates exist for data structured such that

$$(x^+_{min}, x^+_{max}) \cap (x^-_{min}, x^-_{max}) \neq \phi \tag{3}$$

where x^+ and x^- indicate stress levels at which a response and a nonresponse was observed, respectively. This situation is commonly referred to as a zone of "mixed" results. An additional requirement for the algorithm is that there be a minimum of two nonresponses and responses.

AVR, an alternative estimation procedure, is the average of stress levels providing the k lowest responses and the k highest nonresponses. The value k is generally taken to be 3, when possible, but no less than 2 is considered here. AVR is used currently by the Combat Systems Test Activity (CSTA) of APG when their algorithm for maximum likelihood estimation fails to converge or when there is no zone of "mixed" results. This method could be considered an extension to comments made by Brownlee, Hodges, and Rosenblatt in a 1953 paper. There they suggest averaging stress levels which reasonably hold pertinent information for the estimate of the median using data resulting from Up and Down testing. Wetherill (1963) cautions that before an average of stress levels is made, consideration should be given to the type of response elicited by these stress levels. The AVR method mentioned here does consider type of response. It is supported somewhat in the knowledge that a convergent type of algorithm, such as the Langlie, should provide one with stresses about the median.

The final estimation procedure considered is the Next Stress. In the case of the Langlie strategy, it is not expected to be a very good estimator, since it is very possible that this final estimate may consist of the average between a design point and a gate.

In order to compare designs in a real-life environment, reasonable estimation must be possible for each set of test conditions considered. If the NMLE is far and away the best estimate of V_{50} when it exists uniquely but only exists uniquely 50% of the time, much testing could go wasted. In practice, an estimate is usually attempted rather than waste the money and efforts expended during the test, either for direct use or for designing another experiment. For this reason, the estimation procedures actually used will be combinations of the above three methods, thus insuring that some estimate will always be made regardless of test conditions. We refer to these as practical estimation schemes. The practical estimation schemes used in connection with the Langlie will be as follows.

Next Stress is the next design point of the algorithm. There is no need for a back-up procedure if Next Stress is being considered, since a next design point will always be possible using the algorithm.

For AVR, it is thought that a minimum of two responses and nonresponses should be considered when averaging the stress levels. Thus, if there is only one response or nonresponse, a back-up procedure will be used. We have chosen Next Stress as this back up procedure. Hereafter, when considering the technique AVR, we are referring to this two-level process.

In terms of maximum likelihood estimation, the following three-level estimation process will be used. First, use the NMLE's if they exist uniquely within the range of the data. Second, when the NMLE's do not exist uniquely, use the AVR technique. Occasionally the NMLE's will exist uniquely but outside the data range. In our experience this commonly results from a high concentration of observations taken near a gate where, for example, most are nonresponse and a small percentage are response. The NMLE's are then some value above the gate. Many times this estimate is greatly inflated because testing has not been performed in the correct region, that is

not near the median. It is then necessary to consider a back-up estimation procedure. Since the reason for this situation is a poor choice of design levels, AVR is not a reasonable choice. In fact, a reasonable choice may not exist given the restriction of the Langlie gates. We chose to make this estimate Next Stress, which in this situation will approximate the gate. We realize that this convention is not ideal, but first remember that in our experience it has only occurred in the above undesirable situation. Secondly, the primary function of these designs is to provide observations giving rise to good estimation. Failure to do so under varying conditions is simply a penalty of using that design. The number of necessary Next Stress gate approximations will be recorded and considered when we do the final design comparison. Hereafter, when considering the NMLE estimator, we are referring to this three level process.

These three practical estimation procedures will be used for each of the five designs. The only difference in practical estimation among the designs will occur with the designs' choice of Next Stress. This amounts to the comparison of 15 possible data collection and estimation procedures.

2. Delayed Robbins-Monro Stochastic Approximation Method (DRM)

The Stochastic Approximation Method was proposed by Robbins and Monro in 1951. The DRM is essentially this same procedure with an adjustment to help the design sample from the correct region. So that the experimenter will be required to think of the sensitivity test as he is accustomed, the starting position for the DRM and the magnitude (to be discussed later) of the constant c will be be drawn from the same information that the experimenter commonly provides in order to use the Langlie procedure. Select a starting value in the same manner as for the Langlie strategy. Cochran and Davis (1964), and Davis (1969) suggest making design points, prior to a reversal,

$$x_{n+1} = x_n - c(y_n - .5), \tag{4}$$

with y_n, a (0,1) outcome and with c a constant. Reversal is the occurrence of (response, nonresponse) or (nonresponse, response) in succession. Subsequent design points will be chosen according to the usual RM method by

$$x_{n+1} = x_n - \frac{c}{n-k+1}(y_n - .5), \tag{5}$$

where k is the first sample number corresponding to the first reversal. The results of Davis' small sample study show the DRM to be one of the best performers of the designs considered.

3. Adaptive Robbins-Monro (ARM)

The ARM is discussed by Anbar (1978). Wu (1985) suggests an alternative truncation rule which will be used here. As with Langlie and DRM, the ARM will request the experimenter's guesstimate of reasonable upper and lower gates. Design points are picked as

$$x_{n+1} = x_n - c(y_n - .5) \tag{6}$$

until a reversal occurs. It has been shown through the work of Chung (1954) and Hodges and Lehmann (1955) that the optimum c is $(F'(.5))^{-1}$, where F is the response function for the critical velocity. Use of the theoretical result is made when choosing subsequent design points by

$$x_{n+1} = x_n - \frac{\hat{\beta}_n^{-1}}{n-k+1}(y_n - .5), \tag{7}$$

where $\hat{\beta}_n$ is the regression slope estimate given by

$$\hat{\beta}_n = \frac{\sum_{i=1}^{n} y_i(x_i - \bar{x}_n)}{\sum_{i=1}^{n} (x_i - \bar{x}_n)^2}, \tag{8}$$

and k, as before, is the sample number marking the first reversal. Because of the variability commonly associated with $\hat{\beta}$, the ARM actually only uses $\hat{\beta}$ provided that it falls within a specific range. Wu suggests truncating $\hat{\beta}^{-1}$ rather than $\hat{\beta}$. This yields a next design point of

$$x_{n+1} = x_n - \frac{d_n}{n-k+1}(y_n - .5), \tag{9}$$

where d_n is given by

$$d_n = \max(\min(\hat{\beta}_n^{-1}, c), \delta) \tag{10}$$

with $c > \delta > 0$ and c to be discussed later.

4. Estimated Quantal Response Curve (EQRC)

The EQRC proposed by Wu (1985) is a general technique by which the parameters of an assumed response function can be estimated. In this study the logit response function is assumed and maximum likelihood estimation is employed. The next design point is taken to be the logit-MLE (LMLE) when it exists uniquely, based on the data up until that point. This requires that data be collected in an alternate fashion until the LMLE can be used. Based on Wu's suggestion, we choose to use two different base designs: DRM, ARM. EQRC-DRM and EQRC-ARM are treated as separate data collection procedures.

Unique existence of the LMLE, Silvapulle (1981), is guaranteed by a zone of "mixed" results. Even when it exists uniquely, the estimates can vary greatly for small sample sizes. For this reason, the base design is used up to and including sample point six or whenever the LMLE exists uniquely, which ever comes later.

The logit distribution is given by

$$F(x \mid \theta) = \frac{1}{1+e^{-\lambda(x-\alpha)}} \quad \lambda > 0, \ \theta = (\alpha, \lambda). \tag{11}$$

where $\hat{\alpha}$ is the desired estimate for V_{50}. The maximum likelihood estimates can be obtained by solving

$$\sum_{i=1}^{n} F(x_i \mid \alpha, \lambda) = \sum_{i=1}^{n} y_i \tag{12}$$

$$\sum_{i=1}^{n} x_i F(x_i \mid \alpha, \lambda) = \sum_{i=1}^{n} y_i x_i \tag{13}$$

in an iterative fashion. In practice, it is doubtful that field experimenters will draw on an iterative solution to these equations for their next stress level. Realizing this, Wu (1985) suggests using an approximate solution to these equations. The approximation

$$\frac{1}{1+e^{-t}} \approx \frac{1}{2} + \frac{1}{6}t \tag{14}$$

is substituted in the above equations for the logistic distribution, $F(x|\theta)$, defined in Equation 11. Then the next design point $\hat{\alpha}$ is a weighted average of the stress levels expressed by as

$$\hat{\alpha}_n = \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} (y_i - \frac{1}{2}) x_i - \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} (y_i - \frac{1}{2})}{\sum_{i=1}^{n} 1 \sum_{i=1}^{n} (y_i - \frac{1}{2}) x_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} (y_i - \frac{1}{2})}. \tag{15}$$

The accuracy of this approximation varies with $t=\lambda(x-\alpha)$. Cox (1970) determines that the approximation will be within .07 of $\frac{1}{1+e^{-t}}$ for values of t in the interval (-3, 3). The corresponding rule, stated in terms of stimulus location, insures .07 accuracy for $F(x|\theta)$ for stimulus levels, x, within 1.65 standard deviations of the mean. For stimulus levels gathered outside that interval, for example, at two, three, or four standard deviations from the mean, the accuracy degrades quickly to within .13, .41, and .71, respectively. This fact implies that the approximation is poor when the function is evaluated at stimulus levels distant from the mean. However, this problem is not anticipated to be serious for two reasons. First, the convergent nature of the sequential procedure will concentrate most stimulus levels about the mean. Second, the weighted average, $\hat{\alpha}_n$, weights more heavily those stimulus levels close to the mean. Therefore, the approximation should perform reasonably well in this setting, and no attempt will be made to identify or eliminate, from estimation, extreme levels of the stimulus.

Besides insuring that the LMLE be used no sooner than seven samples into the design, it is also desirable to truncate the design thus limiting moves which are too large. With this additional consideration the data collection procedure is as follows for both EQRC-DRM and EQRC-ARM.

Sample using the DRM or ARM procedure until the LMLE exists uniquely and sample six has been taken. Subsequent design points are taken to be

$$x_{n+1} = \frac{d_n^*}{n-k+1}(y_n - .5), \tag{16}$$

where d_n^* is determined by

$$d_n^* = \max(\min(d_n, c), \delta), \tag{17}$$

with $c > \delta > 0$. The value d_n is the solution of

$$x_{n+1} = x_n - \frac{d_n}{n-k+1}(y_n - .5), \tag{18}$$

where x_{n+1} is the LMLE $\hat{\alpha}$ of V_{50}, and k is the sample point marking the first reversal.

III. QUESTIONS OF INTEREST

What effect will sample size have on the distribution of $\hat{V}_{50}$? In practice, for penetrator-against-plate testing, sample sizes are commonly less than 15, a fairly restrictive sample for sensitivity testing. We would like to see what happens to $\hat{V}_{50}$ as we vary small sample sizes.

What is the effect of noise on the distribution of $\hat{V}_{50}$? With the exception of Golub and Grubbs (1956), all of the work to date on these sensitivity designs assume that the next design point intended by the design can be attained. In penetrator-against-plate testing this is not the case. We wish to examine these procedures for their sensitivity to different noise environments.

What is the effect of misplacement error on the distribution of $\hat{V}_{50}$? The experimenter's guesstimate of the quantile location for a newly developed item can often be very much in error. In some situations, the median of the response distribution has been guessed with more than 3σ error, as shown by further experimentation. The parameter σ is taken to be the standard deviation of the response distribution. What procedure will smooth over this poor initial guesstimate?

What is the effect of error in the guesstimate of the response distribution standard deviation? Commonly, gates are established at $V_{50} \pm 3\sigma$. The concern is to insure that the median is within the gates since Langlie's procedure makes no allowance for gate adjustment while testing. It sometimes results in ridiculous gate setting, such as $V_{50} \pm 10\sigma$, which results in several of the first few rounds being wasted. Each design and combined estimation procedure will be assessed for its ability to rebound from mildly poor guesstimates.

How do the procedures respond to different underlying response distributions? If a parametric assumption is made, it has usually been probit or logit in the literature. Davis (1969) considers normal, uniform, and exponential response functions in his small sample study. His results indicate robustness to response distribution for each of the estimation procedures considered. We will look at this question again for completeness in light of our additional considerations.

Which estimation procedure fares best over the wide range of test conditions for each of the designs individually? There is no reason to limit ourselves to the intended estimation associated with each design, especially considering that they make different distribution assumptions. We would like to pick an estimator which performs fairly well over a range of real-life representative conditions.

Which design and estimation procedure fares best over all? This is the most important question to be addressed in this analysis. We would like to conclude this study with a sound recommendation for data collection and estimation, yielding reasonable estimates over a wide range of real-life conditions. We will do this by comparing the best estimation results of each design. Also compared are the intended estimation results for each design.

IV. DESIGN CONSIDERATIONS

1. Factors

Five response functions were considered for the distribution of critical velocity: normal, uniform, exponential, Cauchy, empirical. The first three distributions were given a median 0 and a standard deviation 1. The normal was chosen because the Langlie uses this assumption for estimation, and, similarly, a logit assumption is made for EQRC. The uniform was chosen as an extreme case of the symmetric distributions. It's flat density should slow convergence of the sequential designs. The exponential was chosen since it is a common asymmetric distribution and we wanted to see what problems asymmetry causes in data collection and estimation. The Cauchy distribution was chosen for its heavy tails. Parameters were set so that the quartiles were the same as for a normal(0,1) distribution. We wanted to see if those heavy tails adversely affected the estimation of the median. The main observations and conclusions are to be drawn from these four distributions.

The empirical distribution referred to was formed using real penetrator-against-plate data. It had an estimated median of 157.6 and an estimated standard deviation of 40.8. The data consisted of 79 stress levels with an associated response or nonresponse. Using the "method of reversals" as outlined in Rothman, Alexander, and Zimmerman (1965) we formed the empirical distribution function. For easier simulation, we then smoothed the step function with a third degree polynomial, and forced the right limit to 1 and the left limit to 0 in order to make it an actual distribution function.

The sample sizes (SS) considered are 9, 12, and 15. Until recently, CSTA's sample size strategy was to fire as many as 15 rounds in hopes of getting at least 12 observations; erratic flight necessitates invalidation of some rounds. In 1983, because of excessive costs, samples were cut to as low as 9 in the acceptance testing of some penetrators. It is important to attempt to quantify the information loss going from 15 to 12 and from 12 to 9 samples to insure that the reduction in samples is truly cost efficient.

The initial design points or Starting Values (SV) considered are $V_{50} - 3\sigma$, $V_{50} - 1\sigma$, V_{50}, and $V_{50} + 3\sigma$. V_{50} was chosen as the ideal starting value, and $V_{50} - 1\sigma$ is considered as being only slightly off the ideal starting point. For the Cauchy distribution any quantity defined in terms of σ will be calculated with $\sigma = 1$. The symmetric response functions being used will only consider error to one side of the true median. For asymmetric distributions it is necessary to consider error on both sides of the true median.

Gates for the Langlie will be SV $\pm 1\sigma$, SV $\pm 3\sigma$, and SV $\pm 5\sigma$. They will be denoted GW(1), GW(3), and GW(5), respectively. This should give us a wide enough range so that an indication of poor gate setting, if it has an effect, will be noticeable in the results. In practice, setting of gates is done using the information that the experimenter supplies. This information affects all the designs in the study. In section 4.2 we relate gate width of the Langlie, to the truncation constant c shared by the other designs.

Three types of noise were used: none, symmetric, asymmetric. Most research concerns itself with the ideal 'no noise' situation. We also want to include this common situation so that our results may have application outside local testing. The noise associated with firing velocity is thought to be symmetric and normal-like in behavior based on local experience. Hence when considering symmetric noise we will use a normal density, with mean 0 and standard deviation σ_n. We can also imagine situations in which noise might take on an asymmetric shape. We chose to use an exponential density with median 0 and standard deviation σ_n to represent this type of noise. Choice of the value of σ_n for each of the noise distributions was made as follows.

Ten sets of recent penetrator-against-plate testing was analyzed using maximum likelihood estimation with an assumed normal response function. In each case, the MLE of the median was within the set of data and was considered good. In addition, the MLE of the response distribution standard deviation σ was calculated and approximately corrected for bias, Langlie (1962). The sample standard deviation σ_d of the difference between nominal and actual firing velocity was computed. In the simulation it is desirable to relate the noise standard error to the populations standard error. The pooled results of the ten studies indicated that a representative value for σ_n is $.15\sigma$.

2. Controls

In determining the number of iterations to be used for each set of conditions, we did some preliminary simulation using the Langlie design. We successively tried 100 to 1500 iterations using this algorithm with 12 design observations per trial. We examined changes in maximum likelihood and AVR estimates of the median as well as in the $\hat{V}_{50}$'s root mean square error $\sqrt{\text{MSE}}$. The $\sqrt{\text{MSE}}$ did not really change much after 200 but the mean of the empirical estimate density for both AVR and NMLE stabilized at 700 observations. Thus we chose 700 as the number of iterations to be used.

The random number sequence will be used in the following manner. Each of the five designs use the exact same random number sequence for a given cross of factors. All three estimators within each design and for a given cross of factors are based on design points resulting from this common sequence of random numbers. This arrangement nests estimator within design; estimators will be compared only within a given design. However combinations of different designs and estimators can be compared. Discussion of the random number sequence is included in Appendix A.

Based on Wu's 1985 results and those of preliminary simulations of the EQRC-DRM and EQRC-ARM performed by us, it seemed that some truncation was desirable for EQRC and ARM. However, we wanted to avoid excessive truncation which would limit the potential of the design to make large moves when appropriate. The question we battled with was how much truncation should we expect to have if the designs are operating at their best. Early study showed that a truncation constant of 6σ, more severe than any that Wu considered, really performed very similar to a constant of 20σ in terms of $\sqrt{MSE}$ of $\hat{V}_{50}$ for the cases considered. So we felt it reasonable to consider some small constants for this small sample situation.

Achieving comparability among the five designs requires an equivalence between the Gate Width and the truncation constant, c. Recall that Starting Value and Gate Width or truncation constant comprise the usable prior information. Starting Values present no problem for comparability, but Gate Width and truncation do. Larger values for each encourage a wider range of sampling, and smaller values, a more narrow range. A restriction in sampling range is an advantage provided the range covers the true median. For all but the Langlie a maximum move of $c/2(n\text{-}k+1)$ may be made to collect the $(n+1)$st sample, assuming the first reversal occurred at trial k ($c/2$ with no reversal). The Langlie procedure, with out a monotone nonincreasing step size, can move at most $GW/2$ at any step. The level of restriction for each will be, in a sense, the same if we set $c=GW$.

This restrictive situation has the following implications. First, superior performance among any of the RM type procedures can be directly attributed to the procedure's superior design point selection and not to varying freedom in design point range. Second, well chosen gate widths should give a better idea of good and bad truncation constants for all RM procedures. Third, a practical comparison among the designs is facilitated by the truncation relationship. Note that some gates chosen as factor levels act to limit the flexibility of a design. How much limitation they impose should also be related to Starting Value. The truncation constant is limiting only if it causes the design to use the truncation constant in calculation of the next design point an excessive number of times. Based on the early simulations, 1 or 2 truncations for a sample size of 15 seemed reasonable. Truncation will be taken into account in the analysis if some test conditions cause truncations to be excessive.

3. Measured Variables

The response for this experiment is taken to be the empirical distribution of $\hat{V}_{50}$ as characterized by its mean, $\hat{V}_{50}^{*}$ and its root mean square error, $SQRT(\frac{1}{700}\sum^{700}(\hat{V}_{50} - V_{50})^2)$. The empirical density of the estimators will also be examined. In the following this density will often be denoted "empirical estimate density." Analysis will concentrate on the $\sqrt{MSE}$. We also measured some counters indicating how much truncation (gate calling for Langlie) was taking place during the simulation. In addition we recorded the existence or nonexistence of unique maximum likelihood estimates. All of these measured variables will figure into the observations made.

4. Design Configuration

The design layout for this study can be seen in Table 1, and the method of data collection is outlined in Figure 2. The reasons for our design configuration are as follows. The assumption of normality is commonly made for our day to day data. In addition the parametric procedures discussed here are based on the normal and logistic distributions. Hence most of the observations were taken from this distribution. The asymmetric noise considered is really not thought to apply to our specific problem so we looked at it only in conjunction with the normal distribution. We used the two common noise conditions with each of the other response distributions considered. As the normal, uniform and Cauchy densities are symmetric, we felt it necessary only to consider Starting Values on the lower side of the median. Gate Width and Starting Value combinations which do not cover (within the gates) V_{50} are predictably bad situations. We looked at three such combinations, only for the normal response function, to see how much worse those cases are. In the case of the exponential and empirical response functions, we chose to consider five Starting Value and Gate Width combinations. Three start the design at the median with varying Gate Widths. The remaining two will compare starting the design toward the short tail of the distribution as opposed to the long tail. With this design configuration we will be able to look at all of the items of interest stated previously.

V. ANALYSIS

1. Observation Methods

Because of difficulties associated with applying formal statistical tests to this type of data, analysis is confined to graphical and table summarization of these Monte Carlo results. Each effect is quantified using these summarizations. This amounts to a case by case comparison of values in terms of $\sqrt{MSE}$, $\hat{V}_{50}$, and the empirical density of the estimator with more emphasis being given to $\sqrt{MSE}$. For example, consider Figure 3 containing values of $\sqrt{MSE}$ for each of two estimation procedures over 12 practical situations. The factor of interest discussed here will be noise. We consider method B and A about the same for zero noise, but B is generally superior to A over the range of

TABLE 1.

DESIGN MATRIX

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW	NORMAL NOISE SV & GW	ASYMMETRIC NOISE SV & GW
NORMAL	9,12,15	$(V_{50} - 3\sigma) \pm 1\sigma$ $(V_{50} - 3\sigma) \pm 3\sigma$ $(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50} - 1\sigma) \pm 1\sigma$ $(V_{50} - 1\sigma) \pm 3\sigma$ $(V_{50} - 1\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$	" " " " " " " " "	" " " " " " " " "
UNIFORM	15	$(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50} - 1\sigma) \pm 3\sigma$ $(V_{50} - 1\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$	" " " " " "	NONE
CAUCHY	15	$(V_{50} - 3) \pm 5$ $(V_{50} - 1) \pm 3$ $(V_{50} - 1) \pm 5$ $(V_{50}) \pm 1$ $(V_{50}) \pm 3$ $(V_{50}) \pm 5$	" " " " " "	NONE
EXPONENTIAL	15	$(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$ $(V_{50} + 3\sigma) \pm 5\sigma$	" " " " "	NONE
EMPIRICAL	15	$(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$ $(V_{50} + 3\sigma) \pm 3\sigma$	" " " " "	NONE

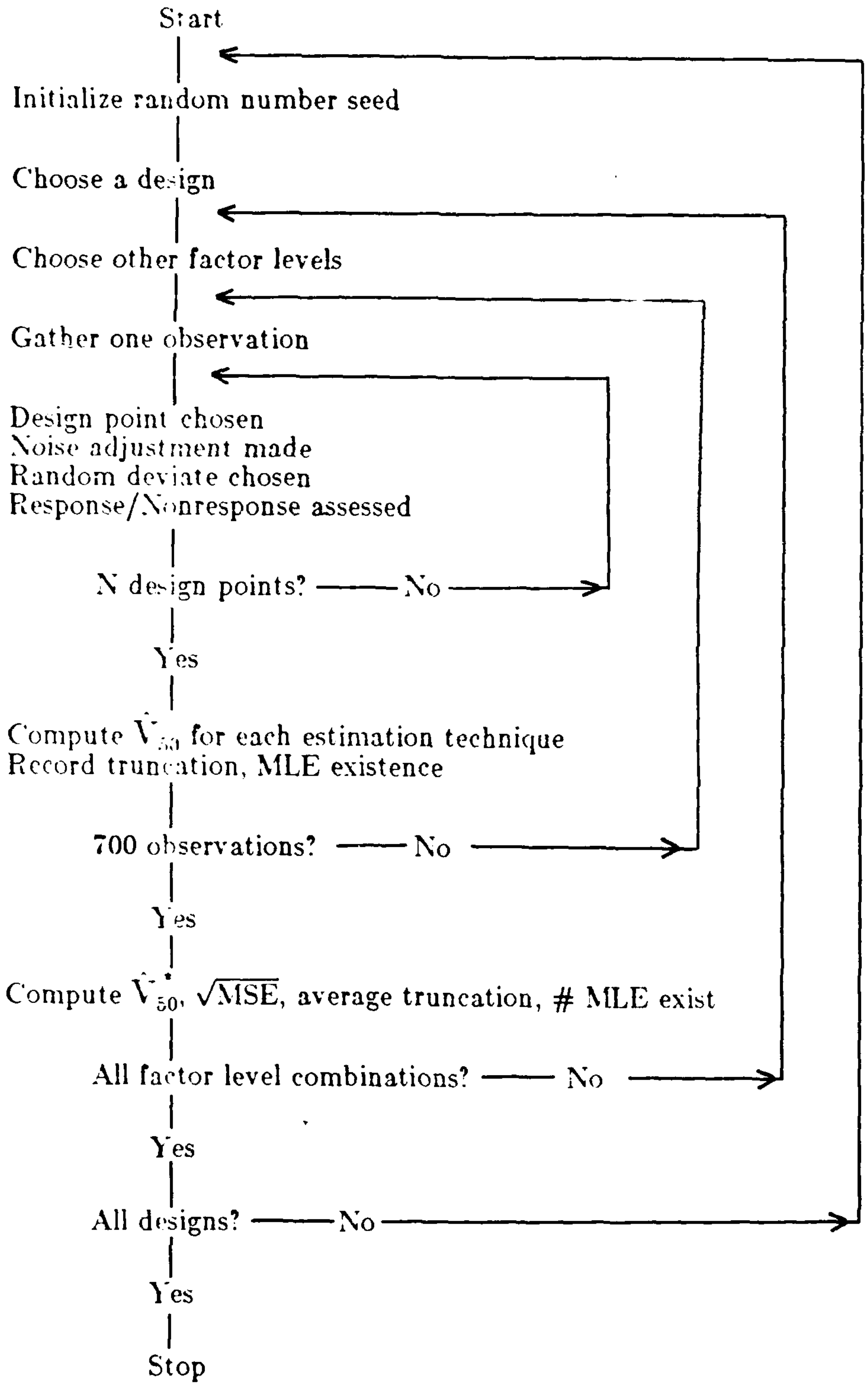

Figure 2. Flowchart of observation collection

normal and asymmetric noise cases. Further B seems relatively insensitive to either type noise, whereas method A generally has a higher $\sqrt{MSE}$ with both normal and asymmetric noise. Case 2 is different in that it always contains the worst showing for method B. Other factor levels would be examined to help explain this.

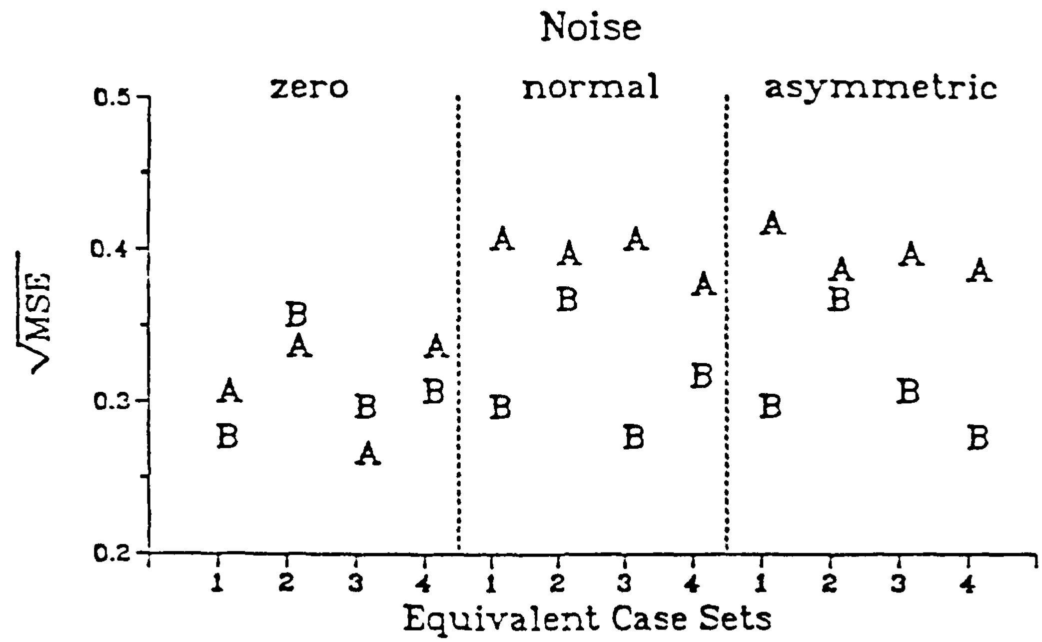

Figure 3. Comparison of two estimation procedures.

This type of graphical study is fairly representative of our approach to making observations. Above, when directly contrasting the two methods, we are looking at two things primarily: the number of times that B was less than A, the magnitude of the difference between B and A. The smallest difference that we are going to consider is $.04\sigma$. The reason for this is that the primary function of this study is to make a recommendation as to design and estimation for our weapons environment. The smallest noise situations that we observed when characterizing σ_n had standard deviations of approximately $.04\sigma$. In our environment, to favor method B's estimate of V_{50} over method A's when A and B differ by less then $.04\sigma$ is not wise as we would be favoring one velocity estimate over another when their difference is within the noise of our firing capability.

One last note concerning our analysis involves the handling of the empirical response function. Similar to an approach in modeling, where one saves a portion of the data to be used as a check against the model, we have chosen to use results from the empirical response function as a check of observations made regarding the other three response functions.

2. Estimate Existence and Truncation

Prior to making observations regarding the factors in the experiment, we will make some statements concerning the measured variables other than the response. For GW(1) over all SV's, the maximum likelihood estimates did not exist uniquely approximately

3% to 15% of the time; for the Langlie, in the cases where the gates did cover V_{50}, they failed to exist uniquely approximately 3% to 24% of the time. Of course, the exception is that when the gates did not cover V_{50} for the Langlie, the MLE failed to exist uniquely almost all of the time. This problem of unique estimates was particularly bad for SV(-3), GW(1) which generally yields the higher percentage. With the wider gate widths, (GW(3), and GW(5), regardless of SV), the MLE exists about 95% of the time. It is interesting that for the normal response function, additional samples only sightly increase the chance of MLE unique existence, thus reminding us of the importance of the beginning samples.

As expected, SV and GW were the determining factors in controlling the amount of truncation which took place. When the gate width was just 1, and the starting value was (-3), the number of EQRC truncations were restrictive, sometimes truncating as much as 60% of the time. In fact, the number of truncations for GW(1) and both SV(-1) and SV(0) is still fairly high. For GW(3) and GW(5) the situation was much better. In the case of EQRC-DRM and EQRC-ARM the average amount of truncation taking place per iteration for sample sizes 9, 12, and 15 was less than 1, less than 2, and approximately 2 respectively. Of the two, GW(5) generally causes slightly less truncation than GW(3).

In the case of ARM, it is also true that GW(1) has a severe effect on the number of truncations made and that GW(3) and GW(5) are much less restrictive. By comparing the number of truncations of ARM to the number of truncations from the ARM portion of EQRC-ARM, we determined that the greatest number of truncations are occurring fairly late in the data collection for ARM.

In the case of the Langlie design, SV(0) regardless of GW, causes the average number of calls to either gate to be between 1 and 2 for all sample sizes. For other SV's, the number of calls to the gates rise considerably. This situation is at its worst when the gates fail to cover the median.

In summary, when comparing designs, it will be necessary to examine most closely those cases in which the number of truncations are reasonable. For this reason, attention will be given to GW(3) and special attention to GW(5) since all the designs seemed to be fairly free to move in those environments. The extent to which the maximum likelihood estimation can be relied upon in combination with the design will also be considered.

3. <u>Sample Size</u>

For the normal response function and all estimation techniques and designs, $\sqrt{\text{MSE}}$ decreases with increasing Sample Size. The precision gain of 6 additional samples in terms of the $\sqrt{\text{MSE}}$ associated with NMLE averages $.13\sigma$ for all the RM type designs and $.10\sigma$ for the Langlie strategy. See Figure 4. In the figures that follow, "case" is meant to mean a SV and GW combination. The precision gain of 6 samples for AVR's $\sqrt{\text{MSE}}$ averages about $.11\sigma$ for RM type designs and $.08\sigma$ for Langlie. Precision gain for Next Stress ranges between $.06\sigma$ and $.12\sigma$. For the normal response function and all

estimation techniques for the RM type designs, increased sample size accentuates the effect of asymmetric noise. This effect, a biasing of $\hat{V}_{50}$, will be discussed later.

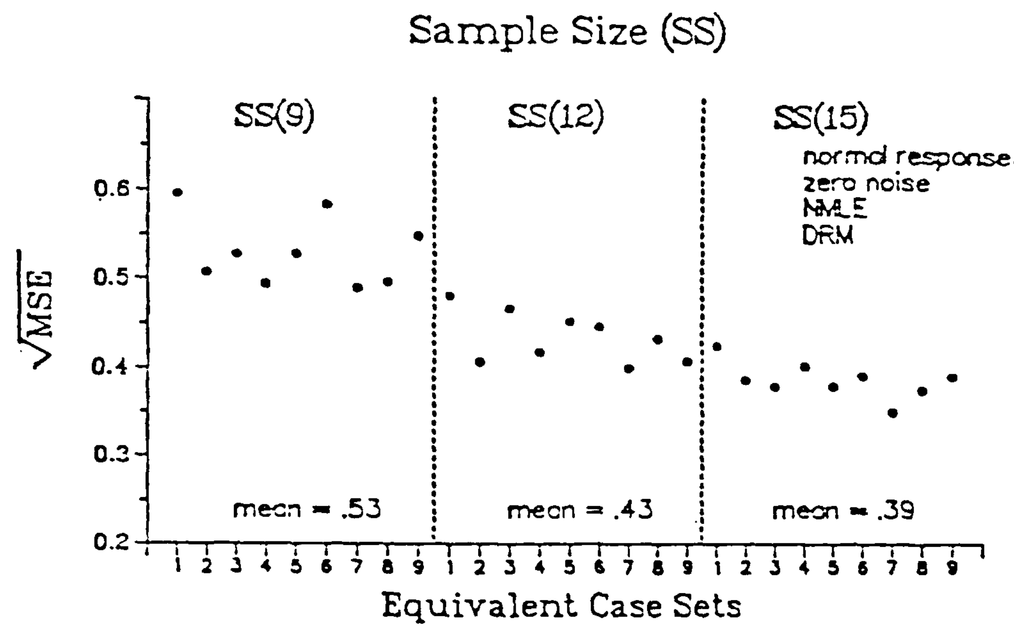

Figure 4. Effect of sample size on $\sqrt{MSE}$.

4. Noise

In general for NMLE, AVR, and Next Stress, zero noise and normal noise seem to produce the same range of estimates for V_{50}, whereas asymmetric noise will drive the estimates toward the upper tail of the response function. Examples are given in Figures 5 and 6. This observation is consistent with the appearance of the empirical densities for all the estimation techniques. There, asymmetric noise causes a slight skewness toward the higher velocities, particularly for narrow Gate Width cases. In addition, asymmetric noise may shift the median away from the known value. Normal and zero noise situations leave an empirical estimate density which is symmetric and centered about zero, the known V_{50}. The exception to this observation is with Langlie which seems relatively insensitive to noise except for the following isolated cases. The first situation is that $\hat{V}_{50}$ for SV(-3) and GW(5) is higher for asymmetric noise than for zero or normal noise. The second is that normal noise acts to drive upward the Next Stress estimate of V_{50} for an exponential response function. The effects of noise on $\hat{V}_{50}$ are more noticeable for the DRM and ARM designs regardless of estimation technique. As stated previously, larger sample sizes make the effects of noise on $\hat{V}_{50}$ more noticeable.

```
                                     SYMBOL   COUNT      MEAN      ST.DEV.
                                       X       700      0.193       0.499
                               EACH SYMBOL REPRESENTS        1 OBSERVATIONS
INTERVAL
NAME            5    10   15   20   25   30   35   40   45   50   55   60   65   70
         +----+----+----+----+----+----+----+----+----+----+----+--------------
*-2.5    +
*-2.4    +
*-2.3    +
*-2.2    +
*-2.1    +
*-2      +
*-1.9    +
*-1.8    +
*-1.7    +
*-1.6    +
*-1.5    +X
*-1.4    +
*-1.3    +
*-1.2    +
*-1.1    +X
*-1      +X
*-.9     +XX
*-.8     +XXXX
*-.7     +XXXXXXXXXXX
*-.6     +XXXXXXXXXXXXXXXXXXXXX
*-.5     +XXXXXXXXXXXXXXXXXXXXXXXXXX
*-.4     +XXXXXXXXXXXXXXXXXXXXXX
*-.3     +XXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.2     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.1     +XXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-134E-9 +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.1      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.2      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.3      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
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*.5      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.6      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.7      +XXXXXXXXXXXXXXXXXXXXXX
*.8      +XXXXXXXXXXXXXXXXXXXX
*.9      +XXXXXXXXXXXXXXXXXXXXXXXX
*1       +XXXXXXXXXXXXXXXXXXXX
*1.1     +XXXXXXXXXXXXXXXX
*1.2     +XXXXXXXXXXX
*1.3     +XXXXXXXXX
*1.4     +XXXX
*1.5     +
*1.6     +X
*1.7     +
*1.8     +
*1.9     +
*2       +XX
*2.1     +
*2.2     +
*2.3     +
*2.4     +
*2.5     +
*LAST    +
         +----+----+----+----+----+----+----+----+----+----+----+----+----+-----
              5    10   15   20   25   30   35   40   45   50   55   60   65   70
```

Figure 5. Effect of asymmetric noise on Next Stress

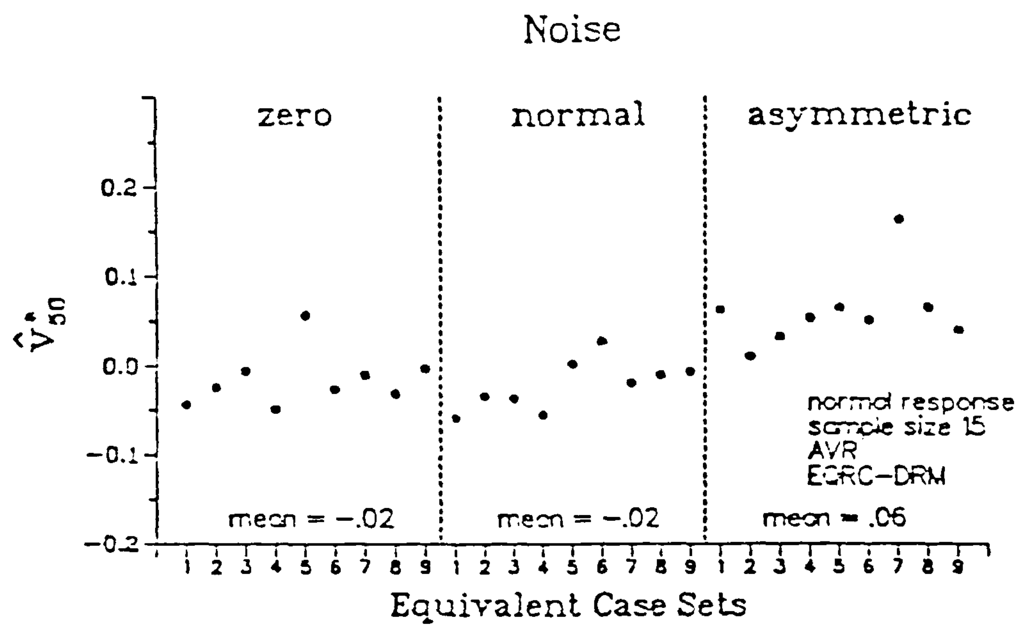

Figure 6. Effect of noise on $\dot{V}_{50}$.

The estimation technique seems to interact with noise for $\sqrt{MSE}$. For each data collection procedure, the range of NMLE and AVR estimates' $\sqrt{MSE}$ is about the same over all noise within a response function and sample size. See Figure 7.

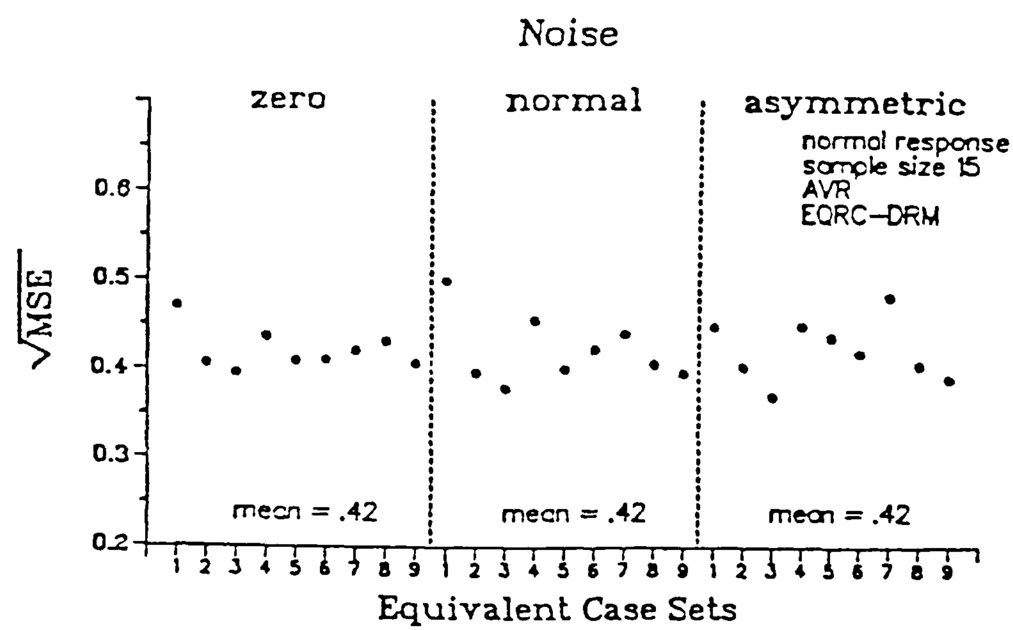

Figure 7. Effect of noise on $\sqrt{MSE}$ for AVR.

However, the $\sqrt{MSE}$ associated with Next Stress inflates some for normal noise and still more for asymmetric noise. This trend is more noticeable for increased samples. It is also more evident for DRM and ARM than for EQRC. This is an important observation since Next Stress is the intended estimation procedure for the RM designs. See Figure 8. For Langlie, stress varies so much that such a claim would be difficult to make.

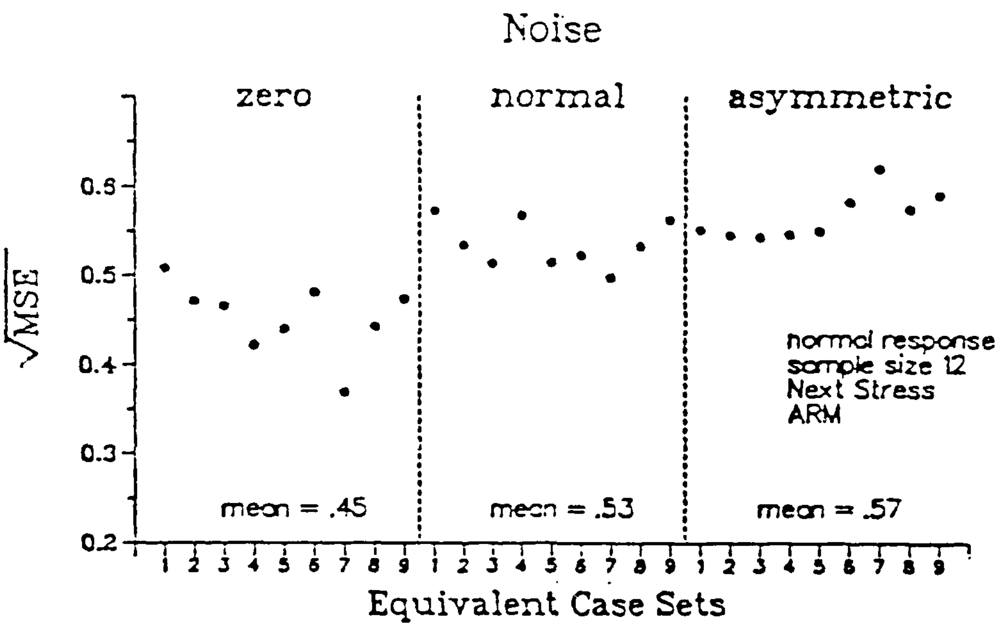

Figure 8. Effect of noise on $\sqrt{MSE}$ for Next Stress.

5. Starting Value and Gate Width

Starting Value and Gate Width interacted with noise, sample size, design, and estimation. As one might expect general rules are difficult to find. The following are some of the notions inferred from observing the results obtained by varying SV and GW.

GW's causing severe truncation usually limit the designs' ability to rebound from a poor decision in SV or a poor decision during the data collection process thus causing higher $\sqrt{MSE}$'s. An exception to this is SV(0) and GW(1) where limited movement forces the design to collect data close to the median.

Among SV's and GW's promoting light truncation, GW(5) seem to produce reasonable results relative to the more narrow gates. For sample size 9, GW(1) and GW(3) were usually better performers than GW(5). However, for 15 samples all the GW's performed similarly. For the exponential response function there was no real difference between SV(-3) and SV(3), with GW(5). This was probably due to GW(5) rather than the designs' indifference to SV for an asymmetric distribution.

SV and GW combinations such that the median was at a gate or outside both gates for the Langlie were predictable. If the median was at the gate the algorithm would begin to converge toward the gate. Thus, the NMLE procedure would appear to be very good in this situation. However, if the median was outside of the gates, it is impossible for a reasonable estimate of the median to be made.

The overriding element in all of these observations is simply, altering the SV and GW will affect the designs' ability to gather information near the median. The design needs to be able to rebound from an incorrect decision (different than "ideal"). For instance, if we consider SV(-3) and GW(2), it is quite possible to be sampling at $V_{50} + 5\sigma$ on the third sample. These kind of wrong decisions are very detrimental in small sample problems.

6. Response Function

For performance over different response functions, we only considered a sample size of 15. In general, over all estimation techniques and designs, normal, uniform and Cauchy response functions yield $\hat{V}^*_{50} \approx 0$; whereas for exponential, $\hat{V}^*_{50} \approx .12$ for RM designs, and .16 for Langlie. This is true of both normal and zero noise situations. One exception seen is $(V_{50} - 3\sigma) \pm 5\sigma$ where occasional outliers for the Cauchy response function lowers the average estimate. See Figure 9.

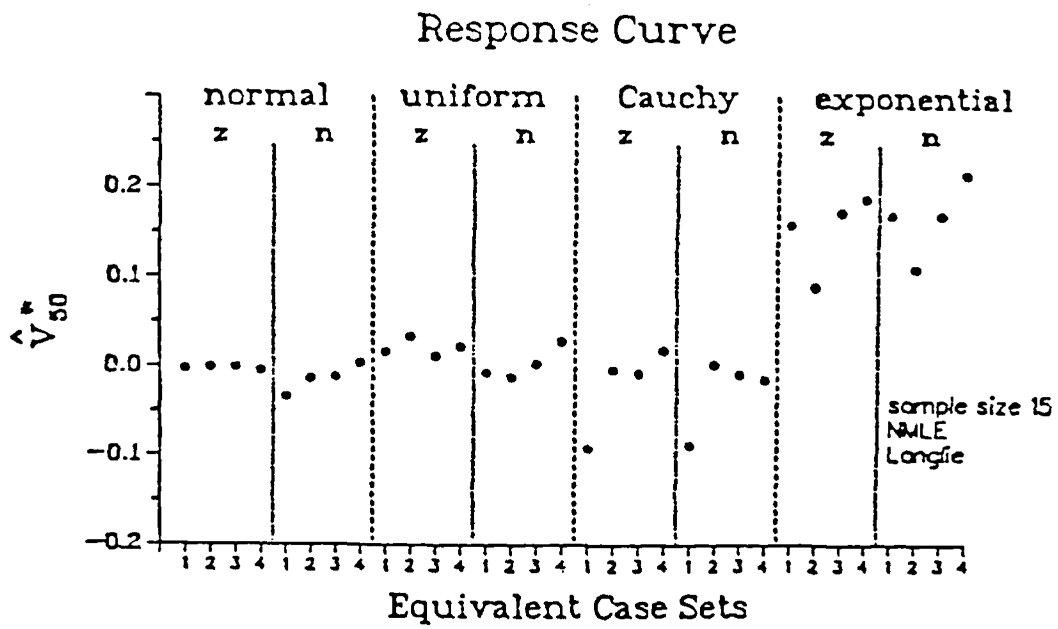

Figure 9. $\hat{V}_{50}$ for four response curves with zero and normal noise.

For the Cauchy distribution, although still centered at zero, we begin to pick up outliers in the empirical estimate density. Outliers as far out as 2.5 from zero as compared to 1.4 for normal were observed. It is easy to imagine why this occurred. With the Cauchy, when sampling in the tail there is a greater probability of an incorrect decision thus sending the sequential algorithm in the wrong direction. This is an important consideration since in practice the underlying response distribution is unknown. On a small sample basis, what might appear to be normal-like even to the point of matching quartiles, may in fact be Cauchy, the heavy tails of which could cause severe estimation problems on occasion.

For NMLE and AVR estimation for each design, the $\sqrt{MSE}$ for the uniform is slightly higher (around $.06\sigma$) than $\sqrt{MSE}$ for the normal over most cases. This also seems to be true occasionally with RM designs and Next Stress but there is too much variability to ascertain a general rule. This is the same observation made by Davis (1969). Even with the outliers, the Cauchy response generally yields $\sqrt{MSE}$'s lower than those of the normal. See Figure 10. This is a function of the amount of probability mass immediately around the median. This point will be addressed in a following plot. Although $\hat{V}^*_{50}$ for the exponential response function is higher, $\sqrt{MSE}$ is generally lower (around $.10\sigma$) than for the uniform response functions.

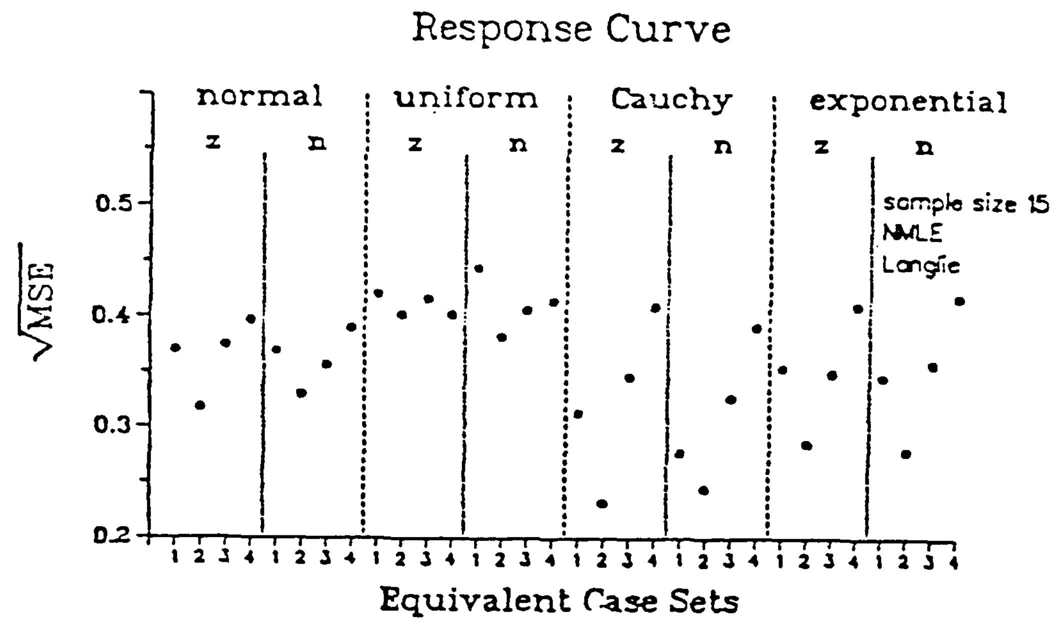

Figure 10. $\sqrt{MSE}$ for four response curves and zero and normal noise.

This is true of all designs and estimators except the Langlie. For the Langlie strategy the difference is more like $.05\sigma$. The primary reason for these response distribution differences is discussed below.

Figure 11 shows the normal, exponential, Cauchy and uniform densities used in this analysis. Comparative evaluation of the distribution functions at particular points about the median will shed some light on the above paragraph. Banerjee (1980) showed that in the case of maximum likelihood estimation of the median of the response function, a concentration of observations about the median would allow for an efficient estimate. It then makes sense that the order of $\sqrt{MSE}$ from lowest to highest was Cauchy, exponential, normal, and uniform since the concentration of probability mass about the median also shares that increasing order. It is also makes sense to expect an estimate biased upwards for the exponential response since the estimates will be either near the median or sometimes much higher. We examined the empirical estimate densities for the various response functions also. Normal and Cauchy response functions yielded fairly peaked and fairly symmetric empirical estimate densities, whereas the uniform yielded a much flatter density but also symmetric. The exponential response caused all empirical estimate densities to be very peaked and skewed toward higher estimates of V_{50}. Normal and Cauchy examples for conditions DRM, NMLE, zero noise, SS(15), and $(V_{50} - 3\sigma) \pm 5\sigma$ are given in Figures 12 and 13. Uniform and exponential examples for the same set of test conditions are given in Figures 14 and 15.

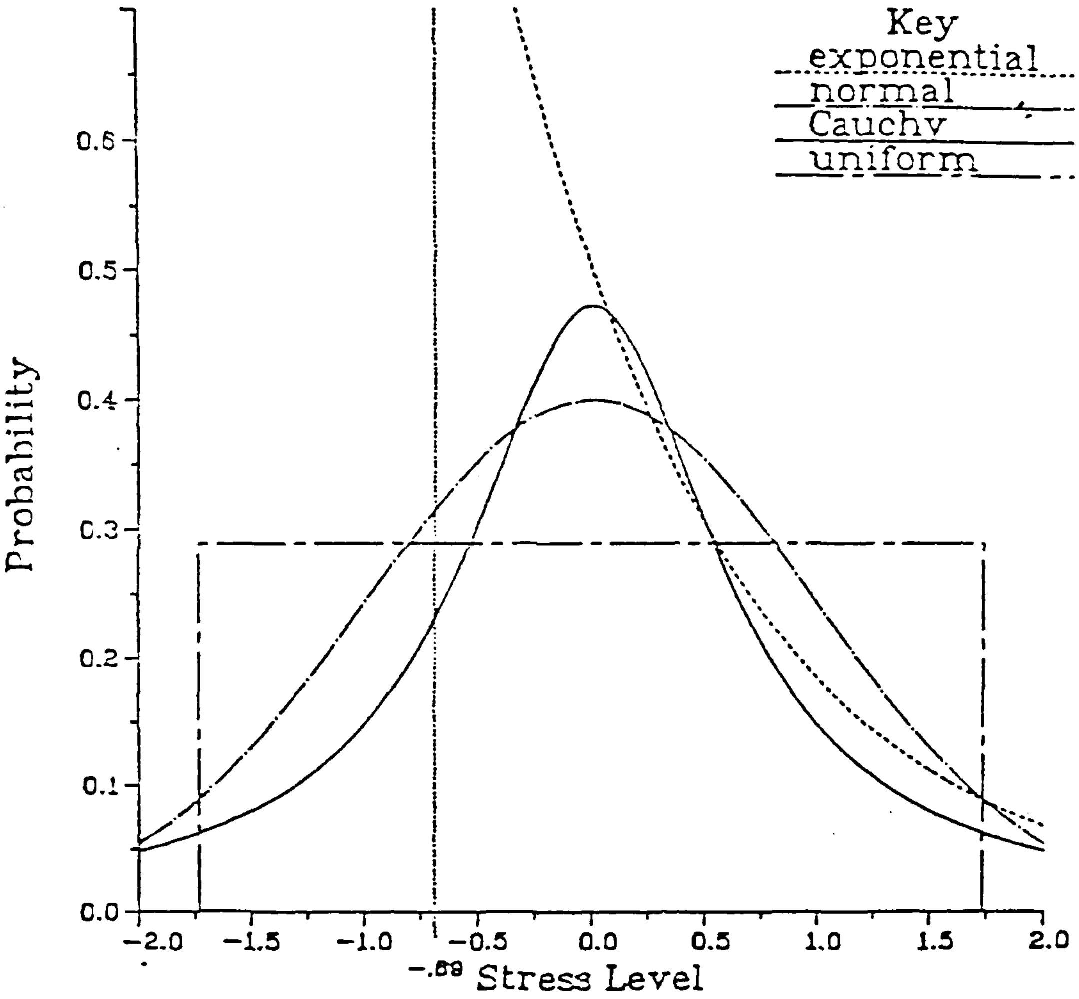

Figure 11. Response function densities

```
                                 SYMBOL  COUNT      MEAN      ST.DEV.
                                   X      700       0.017         0.494
                        EACH SYMBOL REPRESENTS        1 OBSERVATIONS
INTERVAL
NAME            5    10   15   20   25   30   35   40   45   50   55   60   65   70
        +----+----+----+----+----+----+----+----+----+----+----+----+----+----+-----
*-2.5   +X
*-2.4   +
*-2.3   +
*-2.2   +X
*-2.1   +X
*-2     +
*-1.9   +
*-1.8   +
*-1.7   +XX
*-1.6   +
*-1.5   +X
*-1.4   +X
*-1.3   +X
*-1.2   +XXX
*-1.1   +XX
*-1     +XXXXXXX
*-.9    +XX
*-.8    +XXXXXXXXX
*-.7    +XXXXXXXXXXX
*-.6    +XXXXXXXXXXXXXXXXXXXX
*-.5    +XXXXXXXXXXXXXXXXX
*-.4    +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.3    +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.2    +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.1    +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-134E-9 +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX*
*.1     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX*
*.2     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.3     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.4     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.5     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.6     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.7     +XXXXXXXXXXXXXXXXXXXXX
*.8     +XXXXXXXXXXXXXXXXXX
*.9     +XXXXXXXXXXXX
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*1.1    +XXX
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*1.3    +XX
*1.4    +XXX
*1.5    +
*1.6    +
*1.7    +
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*1.9    +
*2      +
*2.1    +
*2.2    +
*2.3    +
*2.4    +
*2.5    +
*LAST   +
        +----+----+----+----+----+----+----+----+----+----+----+----+----+----+-----
             5    10   15   20   25   30   35   40   45   50   55   60   65   70
```

Figure 12. Empirical estimate density for Cauchy response

```
                                   SYMBOL   COUNT      MEAN      ST.DEV.
                                     X       700       0.018        0.379
                         EACH SYMBOL REPRESENTS          1 OBSERVATIONS
INTERVAL
NAME              5    10   15   20   25   30   35   40   45   50   55   60   65   70
         +----+----+----+----+----+----+----+----+----+----+----+----+----+----
*-2.5    +
*-2.4    +
*-2.3    +
*-2.2    +
*-2.1    +
*-2      +
*-1.9    +
*-1.8    +
*-1.7    +
*-1.6    +
*-1.5    +
*-1.4    +
*-1.3    +
*-1.2    +
*-1.1    +X
*-1      +XX
*-.9     +XX
*-.8     +XXX
*-.7     +XXXXXXXXXXXXX
*-.6     +XXXXXXXXXXXXXX
*-.5     +XXXXXXXXXXXXXXXXXXXXXXXXX
*-.4     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.3     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.2     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.1     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-134E-9 +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.1      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX *
*.2      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX *
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*.8      +XXXXXXXXXX
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*1       +XXX
*1.1     +XXX
*1.2     +
*1.3     +X
*1.4     +
*1.5     +
*1.6     +
*1.7     +
*1.8     +
*1.9     +
*2       +
*2.1     +
*2.2     +
*2.3     +
*2.4     +
*2.5     +
*LAST    +
         +----+----+----+----+----+----+----+----+----+----+----+----+----+----+
              5    10   15   20   25   30   35   40   45   50   55   60   65   70
```

Figure 13. Empirical estimate density for normal response

```
                                    SYMBOL  COUNT      MEAN      ST.DEV.
                                      X      700      -0.024       0.438
                          EACH SYMBOL REPRESENTS        1 OBSERVATIONS
INTERVAL
NAME          5    10   15   20   25   30   35   40   45   50   55   60   65   70
         +----+----+----+----+----+----+----+----+----+----+----+----+----+----+
*-2.5    +
*-2.4    +
*-2.3    +
*-2.2    +
*-2.1    +
*-2      +
*-1.9    +
*-1.8    +
*-1.7    +
*-1.6    +
*-1.5    +
*-1.4    +
*-1.3    +
*-1.2    +
*-1.1    +XXX
*-1      +XXXXX
*-.9     +XXXXXX
*-.8     +XXXXXXXXXXX
*-.7     +XXXXXXXXXXXXXXXXX
*-.6     +XXXXXXXXXXXXXXXXXXXXXXXXX
*-.5     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.4     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.3     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.2     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.1     +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX*
*-134E-9 +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.1      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.2      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.3      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.4      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.5      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.6      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.7      +XXXXXXXXXXXXXXXX
*.8      +XXXXXXXXXXXXXXXX
*.9      +XXXXXXXXXX
*1       +XXXXXX
*1.1     +XXXX
*1.2     +
*1.3     +X
*1.4     +
*1.5     +
*1.6     +
*1.7     +
*1.8     +
*1.9     +
*2       +
*2.1     +
*2.2     +
*2.3     +
*2.4     +
*2.5     +
*LAST    +
         +----+----+----+----+----+----+----+----+----+----+----+----+----+----+
              5    10   15   20   25   30   35   40   45   50   55   60   65   70
```

Figure 14. Empirical estimate density for uniform response

```
                                   SYMBOL  COUNT      MEAN     ST.DEV.
                                     X      700       0.147      0.348
                          EACH SYMBOL REPRESENTS        1 OBSERVATIONS
INTERVAL
NAME             5    10    15    20    25    30    35    40    45    50    55    60    65    70
          +----+----+----+----+----+----+----+----+----+----+----+----+----+----
*-2.5     +
*-2.4     +
*-2.3     +
*-2.2     +
*-2.1     +
*-2       +
*-1.9     +
*-1.8     +
*-1.7     +
*-1.6     +
*-1.5     +
*-1.4     +
*-1.3     +
*-1.2     +
*-1.1     +
*-1       +
*-.9      +
*-.8      +
*-.7      +X
*-.6      +X
*-.5      +
*-.4      +XXXXXXXXX
*-.3      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.2      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*-.1      +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX *
*-134E-9  +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX *
*.1       +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX *
*.2       +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.3       +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.4       +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.5       +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.6       +XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
*.7       +XXXXXXXXXXXXXXXXXXXXXXXXX
*.8       +XXXXXXXXXXXXXXXXXXXX
*.9       +XXXXXXXX
*1        +XXXXXXXXX
*1.1      +XXXXXX
*1.2      +
*1.3      +X
*1.4      +X
*1.5      +X
*1.6      +
*1.7      +X
*1.8      +X
*1.9      +
*2        +
*2.1      +
*2.2      +
*2.3      +
*2.4      +
*2.5      +
*LAST     +
          +----+----+----+----+----+----+----+----+----+----+----+----+----+----
                 5    10    15    20    25    30    35    40    45    50    55    60    65    70
```

Figure 15. Empirical estimate density for exponential response

7. Estimation

One important observation is that AVR and NMLE perform virtually the same. There seems to be a slight tendency of NMLE to improve relative to AVR with increased sample size and NMLE's estimate of V_{50} is not affected as much as AVR's for the exponential response function. Furthermore, because of the way AVR is computed. the heavy tails of the Cauchy could cause a very low response or a very high nonresponse to be figured into the average and possibly cause an outlier. Other than for those three situations, the small sample nature of this study makes them about the same. Remember, NMLE's backup procedures include AVR. Still, even in the cases in which NMLE exists about 95% of the time, AVR performs the same. See Figure 16.

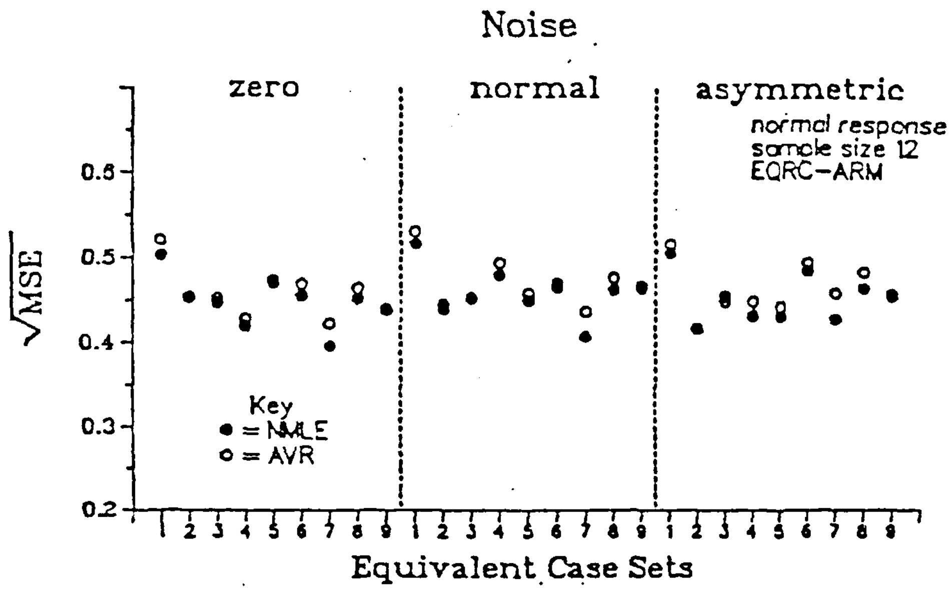

Figure 16. AVR and NMLE estimation subject to noise.

Only under zero noise do Robbins-Monro Next Stress estimates perform consistently as well as AVR and NMLE. Asymmetric noise causes $\hat{V}^*_{50}$ and $\sqrt{MSE}$ to be poorer (higher) for Next Stress relative to NMLE and AVR. This is also true of normal noise when considering only $\sqrt{MSE}$. See Figure 17. Empirical densities show this by yielding a flatter density for Next Stress than for AVR or NMLE. This effect of asymmetric noise is more evident with larger samples and applies only to RM designs. The reason for large sample effect is that there are simply more opportunities for the noise to lead the design away from the median thus causing greater variability. For DRM and ARM we would expect this of small sample problems since noise introduces a greater chance for a wrong decision. For EQRC-DRM and EQRC-ARM and heavy truncation, the Next Stress (a logit MLE) may not be able to get back to the median. However, even in light truncation, the LMLE Next Stress performs slightly worse overall than the NMLE acting on the same design points.

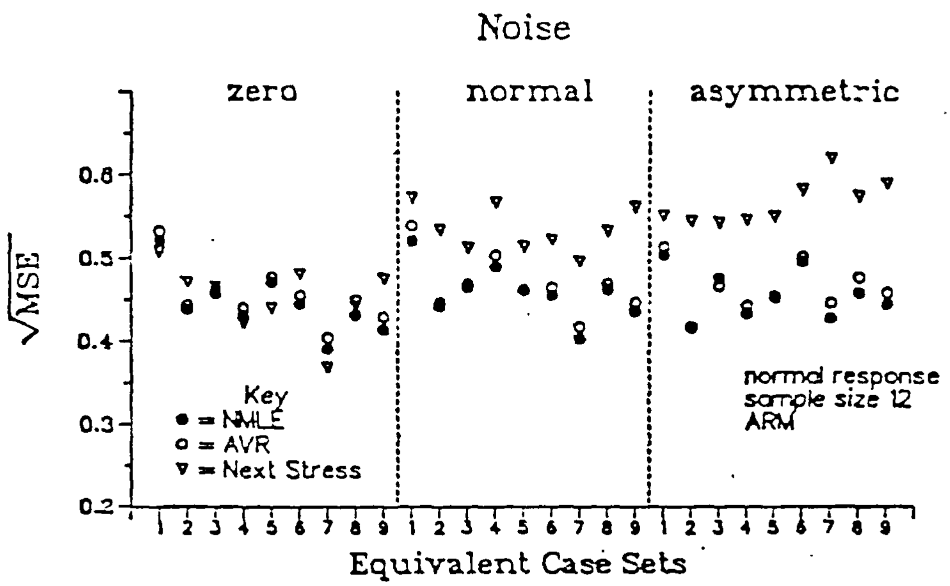

Figure 17. All estimation subject to noise.

There are at least two possibilities why this occurred. One is that for these restrictive samples, the LMLE is not being given enough opportunities to zero in on the median because of the base design. Another is that with small samples, possibly the approximation to logit maximum likelihood estimation is affected by use of design points far from the median.

Over the entire scenario of cases, even for nonnormal response functions, NMLE performs as well or better in terms of $\sqrt{MSE}$ than AVR or Next Stress. Only for the exponential or Cauchy response would we much prefer NMLE over AVR.

8. Design

The designs are compared twice. First, each design with its intended estimation procedure and second, each with its best estimation procedure. Next Stress is intended for the RM designs and NMLE for the Langlie. NMLE was the best estimation procedure for all designs. The ranking of designs with their intended estimation for light truncation where procedures were not severely restricted in design point selection by the truncation constant,

$$\text{Langlie} > \text{EQRC-DRM} = \text{EQRC-ARM} > \text{ARM} > \text{DRM},$$

where $>$ indicates "better than." For heavy truncation where procedures were restricted,

$$\text{EQRC-DRM} = \text{EQRC-ARM} > \text{ARM} = \text{DRM} > \text{Langlie}.$$

Langlie completely reverses its position in our judgement because of its inability to

estimate in some heavy truncation situations. In general Langlie performs better for the uniform response than the others and RM designs are hurt by noise environments.

The final design comparisons takes the best estimator's results associated with each design and compares them among all designs. It turned out that the NMLE yielded the best estimate, given the data gathered by each design. In this situation for light truncation

$$\text{DRM} = \text{ARM} = \text{EQRC-ARM} = \text{EQRC-DRM} = \text{Langlie}.$$

In the presence of heavy truncation,

$$\text{DRM} = \text{ARM} = \text{EQRC-ARM} = \text{EQRC-DRM} > \text{Langlie},$$

since Langlie is again unable to estimate in some heavy truncation situations. Shown in Figure 18 is the $\sqrt{\text{MSE}}$ of each design using NMLE over three GW and SV(0) combinations. These combinations included some light truncation cases shared by all response functions.

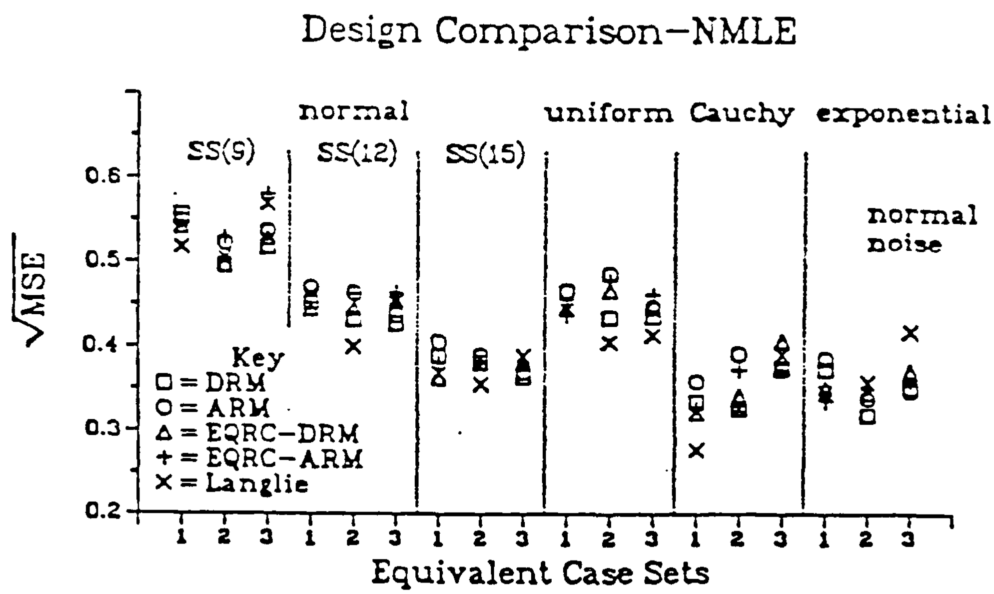

Figure 18. Design comparison.

VI. CONCLUSIONS

As reported by others, we found that poor Starting Values and Gate Widths cause problems in estimating V_{50}. Their primary effects are to limit the number of times that the maximum likelihood estimates exist uniquely and to cause truncation in design point selection to be severe. Both amount to restricted gathering of information near the median, causing poorer estimates. Although it is sensible to say that extremely wide gates will waste the first few rounds fired, we did not demonstrate it here with Gate Width ($\pm 5\sigma$). Apparently, Gate Width ($\pm 5\sigma$) was not wide enough to generally cause wasted rounds.

The response distributions we sampled from did make a difference in maximum likelihood and average estimation. The $\sqrt{MSE}$ associated with each estimate is given from lowest to highest by the ordering: Cauchy and exponential (about the same), normal, and uniform. This order indicates that the median is easier to estimate when the probability mass of a distribution is more dense around the median. Furthermore, the heavy tails of the Cauchy caused outliers in the empirical estimate density. However, a Cauchy distribution and a normal distribution with matching quartiles are not easily distinguishable in the field environment. Experimenters need to be aware of this heavy tail situation particularly where, on the basis of sketchy historical data, normality is assumed. We personally have seen some ballistic data which would fall into this heavy-tailed distribution category.

There is a noticeable difference in the precision of estimators as Sample Size is varied. Changing from sample size 9 to sample size 15 will gain, for example, $\approx .13\sigma$ in precision of the NMLE normal based maximum likelihood estimate with the average of stress levels as a back-up procedure. In fact, since the $\sqrt{MSE}$ for sample size 9 is generally about .52, this change amounts to approximately a 25% increase in precision of the estimate from sample size 9 to sample size 15.

Only asymmetric noise has a great impact on the estimation of V_{50}. Generally it acts to shift the mean of the V_{50}'s upward in varying amounts. However in the case of NMLE and AVR, $\sqrt{MSE}$ is unaffected by noise. In our testing environment this is an important quality.

An important conclusion regarding estimation is that AVR and NMLE are the best performers on the whole over the range of design conditions and that they perform virtually the same in terms of $\sqrt{MSE}$. However, for zero noise situations the intended estimation of the RM type designs, Next Stress, does as well as AVR and NMLE in terms of $\sqrt{MSE}$. Therefore, if there is zero noise in the testing environment and the response distribution is not heavy tailed, there is no particular advantage to using one over the other. In normal or asymmetric noise environments it may be better to use NMLE or AVR instead of Next Stress.

Let us consider the intended estimation of the five designs. We conclude that, among the RM-type designs, Wu's EQRC generally performs as good or better than DRM and ARM over the design scenario. This can be attributed to more efficient design point selection for two reasons. The first is that among the four, all were

restricted to the same maximum step size at each stage of the design. Second, the estimator is the next design point chosen. The fact that Wu's procedure makes a parametric assumption as part of the design point selection does not invalidate this claim since even in the case of the nonnormal response functions EQRC, is still superior. Of the two, EQRC-DRM and EQRC-ARM, there is no difference in performance so we would choose the first because it is easier to carry out.

When given good Starting Values and Gate Widths, the Langlie with intended estimation NMLE will perform only slightly better than EQRC-DRM's Next Stress. However, poor Starting Values or Gate Widths are disastrous for the Langlie. Hence, unless sure of covering the median with reasonably spread gates (a doubtful situation many times in practice) it is better to go with the more flexible EQRC-DRM.

One of the most important things learned from examination of possible estimators in this small sample environment is that the NMLE and AVR methods of estimation fare best even when the response function is nonnormal. It is then reasonable to apply either of these two methods of estimation to data gathered by any of the designs considered provided the distribution is not heavy tailed. With NMLE as the estimator, there is really no difference among the five designs when the Starting Value and Gate Widths are chosen well. This claim is made since there is no one design which is consistently better than the others. However, if Starting Values and Gate Widths are not chosen well the RM designs exceed the performance of the Langlie.

Therefore, we recommend using the Delayed Robbins-Monro with the three level estimation technique NMLE as outlined in the second chapter since it is the easiest to implement among the four RM designs. We would suggest using this with a guesstimate of $V_{50} \pm 3\sigma$. Its primary advantage over the current Langlie procedure is its ability to sample in the correct region despite poor initial estimates. A second advantage is that the NMLE exists uniquely slightly more often for DRM than for Langlie.

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APPENDIX A

MONTE CARLO METHODS

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MONTE CARLO METHODS

Basic to all simulation studies is a good uniform random number generator. It is really up to each individual to establish what good is. Their definition should be dependent on the problem with which they are faced. Different problems require emphasis on different acceptance criteria usual for uniform (0,1) random numbers. We have set forth here what we consider to be the important aspects of random numbers in our study and how well the random numbers measure up to these standards.

First, we wish our random numbers to pass some goodness of fit criterion. We chose a chi-square test with bins of width .025. We generated 500,000 random numbers, stopping to calculate the chi-square test statistic after each 1000 numbers. Testing at the $\alpha = .10$ level, only 7 times, ranging from 16,000 to 161,000 were we able to reject the uniform hypothesis. At no time could we reject at the .05 level. Note that it is not true that we should have necessarily observed 50 and 25 rejections for the .10 and .05 levels respectively since we are not talking about separate random samples for each test in this situation. (We have cumulative samples.) Hence, based on goodness of fit criteria, we are pleased with the performance of the random number generator.

Although it is important that large sample results be good, more important to our study is the small sample behavior of the random numbers. Along these lines, there is a certain number of runs of length 1, length 2, ... , that one would expect to observe as well as a certain number of total runs that one would expect. These expectations are listed by Rubinstein (1981). We computed the empirical distribution of the number of runs and compared it with the theoretical distribution. We found no evidence to suggest that the empirical distribution differed significantly from the theoretical distribution.

Of course no random number generator should repeat its seed. This one does not repeat, at least up to the 500,000 numbers that we considered.

Looking at random numbers would not be complete without plots. We plotted empirical densities, bivariate scatter plots, etc., and found no patterns.

Random deviate generation was performed in the usual manner. The uniform (0,1) generation is handled by a system routine. We wished to generate uniform, normal, and exponential random deviates each with 0 median and standard deviation 1. In addition, a Cauchy distribution was considered with 0 median and quartiles matching a normal (0,1) distribution. Normal (0,1) deviate generation was accomplished with the Box-Muller results. The uniform $(-\sqrt{3}, \sqrt{3})$, exponential (1, -ln2) and Cauchy (0, .675), were generated by setting their respective distribution functions equal to the random number, r, and then solving for the random deviate as a function of r. In the case of the empirical distribution, we took 79 data points, formed an empirical distribution using the 'method of reversals' and then smoothed it with a polynomial to facilitate simulation. We drew our random deviate from this function.

Noise generation was accomplished similarly to the above. In either the normal or exponential case we had a distribution with 0 median and standard deviation .15. We simply added this random noise deviate to the design point under consideration before comparing it to the random deviate.

APPENDIX B

DATA

APPENDIX B

DATA

We have included in Tables B2 - B16 the $\sqrt{MSE}$ for each set of test conditions. Each table gives the results of crossing all test condition factors for one design and one estimation. The presentation of the observations corresponds to the format of the design matrix repeated from the fourth chapter and given here in Table B1. For example, in Table B2, .595, .480, .424 correspond to sample sizes of 9, 12, 15 respectively for a normal response function, zero noise, and $(V_{50} - 3\sigma) \pm 1\sigma$ as the representation of the experimenter's guesstimate. Also in Table B2, .464 corresponds to a sample size of 15 for a uniform response function, normal noise, and $(V_{50} - 3\sigma) \pm 5\sigma$ as the representation of the of the experimenter's guesstimate.

TABLE B1.

DESIGN MATRIX

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW	NORMAL NOISE SV & GW	ASYMMETRIC NOISE SV & GW
NORMAL	9,12,15	$(V_{50} - 3\sigma) \pm 1\sigma$ $(V_{50} - 3\sigma) \pm 3\sigma$ $(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50} - 1\sigma) \pm 1\sigma$ $(V_{50} - 1\sigma) \pm 3\sigma$ $(V_{50} - 1\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$	" " " " " " " " "	" " " " " " " " "
UNIFORM	15	$(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50} - 1\sigma) \pm 3\sigma$ $(V_{50} - 1\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$	" " " " " "	NONE
CAUCHY	15	$(V_{50} - 3) \pm 5$ $(V_{50} - 1) \pm 3$ $(V_{50} - 1) \pm 5$ $(V_{50}) \pm 1$ $(V_{50}) \pm 3$ $(V_{50}) \pm 5$	" " " " " "	NONE
EXPONENTIAL	15	$(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$ $(V_{50} + 3\sigma) \pm 5\sigma$	" " " " "	NONE
EMPIRICAL	15	$(V_{50} - 3\sigma) \pm 5\sigma$ $(V_{50}) \pm 1\sigma$ $(V_{50}) \pm 3\sigma$ $(V_{50}) \pm 5\sigma$ $(V_{50} + 3\sigma) \pm 3\sigma$	" " " " "	NONE

TABLE B2.

DESIGN (DRM) AND ESTIMATION (NMLE)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
		.595	.480	.424	.531	.496	.456	.537	.480	.430
		.507	.406	.385	.504	.431	.375	.542	.435	.391
		.527	.466	.378	.555	.452	.389	.568	.456	.375
		.494	.417	.401	.504	.476	.430	.483	.475	.418
NORMAL	9,12,15	.527	.451	.378	.534	.436	.378	.518	.429	.406
		.582	.445	.390	.554	.447	.401	.564	.467	.386
		.489	.399	.349	.498	.429	.402	.481	.467	.446
		.495	.432	.374	.496	.432	.382	.518	.421	.375
		.547	.406	.389	.517	.427	.365	.518	.421	.370
		.438			.464					
		.447			.448					
		.452			.442					
UNIFORM	15	.438			.458					
		.440			.434					
		.444			.436					
		.339			.335					
		.329			.341					
		.377			.443					
CAUCHY	15	.297			.333					
		.339			.327					
		.345			.375					
		.374			.373					
		.329			.363					
EXPONENTIAL	15	.313			.319					
		.344			.352					
		.341			.349					
		17.010			17.450					
		17.290			17.350					
EMPIRICAL	15	17.470			17.230					
		16.340			17.830					
		17.350			17.870					

TABLE B3.

DESIGN (ARM) AND ESTIMATION (NMLE)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
		.641	.521	.460	.560	.522	.493	.566	.504	.420
		.540	.439	.399	.517	.443	.375	.537	.417	.385
		.513	.458	.395	.548	.469	.406	.565	.476	.414
		.471	.431	.405	.482	.490	.445	.455	.434	.391
NORMAL	9,12,15	.522	.471	.394	.544	.462	.399	.528	.453	.438
		.597	.446	.386	.567	.456	.400	.577	.496	.418
		.448	.391	.348	.439	.404	.383	.460	.429	.428
		.520	.432	.377	.522	.463	.390	.544	.459	.430
		.568	414	.377	.536	.437	.371	.543	.446	.381
		.452			.467					
		.476			.473					
		.475			.480					
UNIFORM	15	.424			.449					
		.443			.486					
		.452			.447					
		.341			.359					
		.338			.311					
		.396			.378					
CAUCHY	15	.270			.293					
		.321			.392					
		.387			.374					
		.371			.386					
		.324			.342					
EXPONENTIAL	15	.338			.339					
		.342			.348					
		.358			.374					
		17.940			17.940					
		16.640			16.390					
EMPIRICAL	15	17.800			18.750					
		16.600			17.710					
		18.220			18.220					

TABLE B4.

DESIGN (EQRC-DRM) AND ESTIMATION (NMLE)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
		.602	.501	.452	.545	.522	.482	.547	.521	.440
		.502	.422	.406	.507	.434	.377	.544	.428	.402
		.515	.458	.381	.541	.444	.363	.540	.457	.362
		.498	.423	.404	.510	.484	.428	.490	.475	.422
NORMAL	9,12,15	.539	.456	.381	.541	.442	.371	.523	.415	.404
		.574	.451	.389	.574	.454	.405	.563	.467	.396
		.501	.411	.373	.498	.436	.391	.486	.463	.436
		.496	.442	.404	.495	.445	.381	.526	.446	.378
		.544	.440	.401	.528	.455	.382	.525	.448	.374
		.431			.446					
		.454			.466					
		.481			.455					
UNIFORM	15	.458			.467					
		.460			.467					
		.476			.451					
		.339			.322					
		.339			.346					
		.401			.447					
CAUCHY	15	.307			.326					
		.369			.342					
		.374			.407					
		.352			.352					
		.341			.361					
EXPONENTIAL	15	.339			.346					
		.354			.371					
		.335			.345					
		17.010			16.430					
		17.990			17.170					
EMPIRICAL	15	18.010			18.250					
		17.400			18.930					
		17.460			17.230					

TABLE B5.

DESIGN (EQRC-ARM) AND ESTIMATION (NMLE)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	.627	.503	.459	.559	.516	.475	.557	.505	.411
		.546	.453	.410	.530	.439	.373	.533	.416	.391
		.510	.447	.383	.547	.452	.382	.544	.454	.367
		.476	.420	.409	.490	.479	.429	.469	.432	.385
		.525	.470	.375	.537	.449	.381	.514	.431	.400
		.597	.455	.394	.573	.464	.411	.587	.485	.412
		.463	.396	.356	.448	.407	.369	.463	.428	.409
		.521	.451	.396	.528	.462	.382	.539	.464	.409
		.572	.438	.392	.580	.463	.378	.547	.455	.391
UNIFORM	15	.443			.437					
		.469			.471					
		.488			.469					
		.445			.452					
		.468			.479					
		.482			.462					
CAUCHY	15	.308			.329					
		.339			.307					
		.391			.401					
		.281			.295					
		.344			.373					
		.389			.373					
EXPONENTIAL	15	.344			.336					
		.335			.335					
		.352			.351					
		.339			.358					
		.355			.370					
EMPIRICAL	15	17.240			17.420					
		17.470			16.490					
		18.590			18.840					
		17.330			18.080					
		17.290			17.370					

TABLE B6.

DESIGN (LANGLIE) AND ESTIMATION (NMLE)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	2.003 .435 .504 .381 .477 .525 .392 .496 .548	2.000 .359 .444 .313 .424 .463 .349 .431 .429	2.000 .326 .370 .288 .369 .417 .318 .374 .397	2.001 .423 .517 .365 .472 .526 .365 .504 .568	1.994 .355 .463 .324 .435 .455 .361 .400 .452	2.000 .322 .369 .297 .373 .424 .330 .356 .390	1.951 .400 .531 .365 .507 .531 .381 .493 .539	1.943 .332 .443 .342 .415 .483 .370 .424 .445	1.941 .303 .387 .288 .381 .415 .336 .371 .401
UNIFORM	15	.420 .412 .410 .402 .416 .402			.444 .418 .439 .382 .406 .414					
CAUCHY	15	.313 .322 .356 .232 .346 .409			.277 .305 .373 .245 .327 .391					
EXPONENTIAL	15	.354 .286 .349 .410 .387			.346 .279 .358 .418 .378					
EMPIRICAL	15	17.330 15.540 16.750 16.220 16.420			17.030 15.030 15.720 17.810 17.020					

TABLE B7.

DESIGN (DRM) AND ESTIMATION (AVR)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
		.587	.486	.440	.526	.501	.476	.530	.480	.435
		.512	.412	.378	.513	.432	.379	.537	.432	.394
		.502	.465	.390	.533	.453	.398	.565	.457	.387
		.486	.430	.431	.497	.487	.455	.471	.482	.446
NORMAL	8,12,15	.522	.459	.399	.525	.444	.400	.505	.434	.429
		.580	.443	.395	.543	.449	.402	.556	.475	.395
		.490	.427	.395	.489	.447	.434	.483	.491	.479
		.499	.439	.375	.495	.434	.385	.520	.427	.390
		.553	.405	.378	.523	.428	.367	.523	.412	.369
			.431			.463				
			.460			.455				
			.455			.451				
UNIFORM	15		.481			.494				
			.438			.439				
			.423			.417				
			.401			.404				
			.422			.401				
			.483			.577				
CAUCHY	15		.336			.361				
			.469			.361				
			.377			.459				
			.399			.394				
			.369			.394				
EXPONENTIAL	15		.335			.342				
			.355			.354				
			.371			.375				
			17.310			17.680				
			18.910			18.820				
EMPIRICAL	15		17.390			17.430				
			16.390			17.570				
			17.380			18.020				

TABLE B8.

DESIGN (ARM) AND ESTIMATION (AVR)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	.647	.532	.488	.563	.539	.517	.570	.514	.429
		.560	.444	.405	.531	.447	.395	.546	.418	.407
		.497	.463	.412	.532	.465	.428	.546	.467	.429
		.476	.440	.424	.484	.504	.468	.451	.443	.410
		.492	.477	.406	.531	.462	.428	.509	.454	.458
		.587	.455	.400	.550	.465	.420	.549	.502	.442
		.445	.404	.367	.439	.418	.410	.457	.447	.459
		.518	.449	.415	.519	.469	.409	.541	.477	.454
		.578	.429	.383	.540	.447	.391	.543	.459	.392
UNIFORM	15	.455			.482					
		.497			.491					
		.483			.494					
		.446			.477					
		.472			.503					
		.446			.457					
CAUCHY	15	.371			.377					
		.365			.362					
		.445			.444					
		.293			.322					
		.373			.421					
		.457			.447					
EXPONENTIAL	15	.397			.396					
		.346			.361					
		.370			.367					
		.370			.371					
		.387			.406					
EMPIRICAL	15	18.400			18.940					
		17.280			17.550					
		19.040			19.790					
		17.260			18.130					
		18.220			18.570					

TABLE B9.

DESIGN (EQRC-DRM) AND ESTIMATION (AVR)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
		.595	.508	.469	.539	.526	.498	.540	.520	.447
		.518	.436	.406	.520	.446	.396	.539	.439	.402
		.501	.458	.395	.529	.438	.378	.538	.448	.369
		.493	.439	.436	.504	.497	.454	.483	.486	.448
NORMAL	9,12,15	.530	.468	.409	.533	.452	.400	.510	.427	.436
		.568	.460	.410	.561	.465	.422	.550	.486	.418
		.499	.440	.421	.493	.460	.439	.487	.491	.481
		.502	.463	.431	.498	.460	.405	.532	.462	.405
		.552	.440	.406	.527	.459	.395	.528	.445	.389
		.423			.452					
		.473			.474					
		.489			.480					
UNIFORM	15	.503			.509					
		.477			.491					
		.466			.447					
		.410			.389					
		.447			.421					
		.537			.607					
CAUCHY	15	.349			.358					
		.494			.397					
		.440			.508					
		.381			.382					
		.389			.394					
EXPONENTIAL	15	.387			.385					
		.373			.385					
		.366			.381					
		17.080			17.140					
		19.700			19.060					
EMPIRICAL	15	18.940			19.200					
		17.530			19.050					
		17.370			17.330					

TABLE B10.

DESIGN (EQRC-ARM) AND ESTIMATION (AVR)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
		.633	.520	.482	.561	.530	.501	.559	.515	.420
		.550	.453	.407	.539	.444	.375	.535	.416	.401
		.497	.452	.397	.536	.451	.401	.539	.448	.381
		.480	.428	.431	.491	.493	.452	.463	.448	.412
NORMAL	9,12,15	.502	.473	.396	.527	.457	.407	.504	.442	.434
		.582	.468	.411	.553	.469	.428	.563	.494	.431
		.464	.422	.394	.455	.436	.415	.465	.458	.452
		.520	.463	.424	.526	.476	.404	.538	.483	.429
		.585	.439	.395	.581	.466	.391	.546	.457	.395
			.431			.442				
			.493			.483				
			.494			.488				
UNIFORM	15		.482			.487				
			.490			.503				
			.481			.467				
			.354			.369				
			.409			.382				
			.486			.525				
CAUCHY	15		.316			.329				
			.436			.414				
			.459			.449				
			.377			.362				
			.373			.367				
EXPONENTIAL	15		.389			.383				
			.362			.377				
			.381			.390				
			17.340			17.850				
			18.630			18.100				
EMPIRICAL	15		19.690			19.830				
			17.690			18.300				
			17.380			17.700				

TABLE B11.

DESIGN (LANGLIE) AND ESTIMATION (AVR)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	2.003	2.000	2.000	2.001	1.994	2.000	1.952	1.945	1.942
		.445	.371	.348	.437	.377	.342	.412	.345	.318
		.505	.444	.375	.508	.459	.366	.523	.439	.387
		.395	.330	.310	.382	.346	.318	.369	.345	.302
		.466	.427	.382	.466	.437	.379	.496	.420	.392
		.514	.466	.432	.513	.454	.438	.519	.482	.423
		.391	.344	.313	.358	.345	.324	.369	.361	.332
		.495	.438	.374	.501	.400	.362	.493	.425	.379
		.547	.432	.397	.567	.456	.391	.537	.446	.401
UNIFORM	15	.416 .417 .453 .396 .420 .402			.439 .427 .449 .365 .405 .417					
CAUCHY	15	.293 .321 .371 .228 .354 .405			.263 .295 .392 .237 .332 .381					
EXPONENTIAL	15	.361 .279 .358 .416 .391			.358 .272 .366 .421 .379					
EMPIRICAL	15	17.190 15.090 16.710 16.480 16.250			17.010 14.470 16.150 17.700 16.860					

TABLE B12.

DESIGN (DRM) AND ESTIMATION (NEXT STRESS)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	.580	.458	.409	.562	.537	.516	.590	.564	.583
		.536	.449	.439	.563	.513	.479	.609	.507	.509
		.663	.595	.495	.694	.585	.542	.732	.608	.556
		.474	.396	.385	.532	.523	.514	.573	.592	.602
		.533	.475	.399	.582	.496	.452	.574	.528	.501
		.663	.579	.513	.679	.571	.532	.682	.607	.556
		.460	.381	.332	.533	.500	.490	.540	.623	.665
		.544	.471	.418	.559	.507	.474	.602	.523	.498
		.698	.545	.521	.688	.590	.531	.657	.597	.554
UNIFORM	15	.582			.605					
		.457			.534					
		.558			.564					
		.418			.559					
		.480			.513					
		.591			.602					
CAUCHY	15	.375			.401					
		.335			.379					
		.416			.485					
		.288			.405					
		.399			.369					
		.397			.444					
EXPONENTIAL	15	.476			.462					
		.319			.465					
		.362			.415					
		.482			.507					
		.472			.471					
EMPIRICAL	15	21.510			24.340					
		16.520			21.720					
		18.920			20.850					
		21.430			23.860					
		22.260			23.019					

TABLE B13.

DESIGN (ARM) AND ESTIMATION (NEXT STRESS)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	.616	.508	.451	.597	.572	.556	.574	.551	.521
		.585	.471	.442	.585	.534	.493	.626	.545	.532
		.525	.466	.382	.590	.513	.482	.632	.543	.560
		.459	.422	.392	.532	.567	.554	.505	.546	.559
		.480	.440	.373	.564	.514	.484	.586	.550	.569
		.610	.481	.424	.612	.522	.479	.636	.582	.561
		.412	.369	.326	.506	.497	.509	.546	.620	.704
		.522	.442	.394	.582	.533	.490	.628	.574	.580
		.643	.474	.453	.639	.562	.499	.657	.590	.533
UNIFORM	15	.455			.529					
		.455			.559					
		.504			.550					
		.401			.574					
		.454			.568					
		.531			.565					
CAUCHY	15	.311			.373					
		.324			.356					
		.361			.369					
		.257			.391					
		.308			.427					
		.392			.422					
EXPONENTIAL	15	.369			.423					
		.307			.475					
		.358			.420					
		.437			.459					
		.366			.440					
EMPIRICAL	15	17.920			21.360					
		15.640			22.210					
		18.270			22.440					
		19.640			22.920					
		18.700			20.940					

TABLE B14.

DESIGN (EQRC-DRM) AND ESTIMATION (NEXT STRESS)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
		.596	.489	.446	.571	.543	.512	.603	.568	.545
		.518	.451	.432	.526	.454	.412	.569	.448	.430
		.566	.439	.371	.541	.435	.358	.550	.452	.365
		.477	.402	.385	.519	.504	.467	.543	.527	.509
NORMAL	9.12.15	.517	.435	.358	.528	.432	.365	.510	.416	.412
		.587	.520	.422	.616	.493	.499	.621	.506	.454
		.483	.401	.362	.517	.471	.442	.529	.555	.569
		.537	.474	.427	.537	.487	.421	.584	.499	.420
		.626	.521	.486	.637	.550	.485	.617	.553	.460
		.430			.437					
		.442			.474					
		.512			.486					
UNIFORM	15	.446			.513					
		.489			.495					
		.565			.531					
		.342			.319					
		.332			.355					
		.414			.469					
CAUCHY	15	.300			.362					
		.411			.363					
		.401			.431					
		.348			.336					
		.339			.410					
EXPONENTIAL	15	.374			.387					
		.440			.465					
		.348			.346					
		16.740			16.280					
		17.510			19.400					
EMPIRICAL	15	19.070			19.440					
		20.440			22.380					
		17.380			16.550					

TABLE B15.

DESIGN (EQRC-ARM) AND ESTIMATION (NEXT STRESS)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	.597	.490	.454	.574	.534	.509	.567	.548	.492
		.555	.474	.442	.559	.464	.399	.565	.463	.437
		.495	.437	.366	.548	.443	.377	.549	.452	.369
		.460	.401	.391	.507	.502	.465	.489	.473	.458
		.508	.452	.355	.533	.444	.381	.534	.453	.409
		.630	.493	.429	.612	.508	.452	.648	.518	.464
		.448	.387	.348	.487	.445	.421	.520	.537	.566
		.528	.475	.422	.566	.505	.423	.584	.510	.445
		.632	.514	.484	.653	.559	.459	.631	.551	.472
UNIFORM	15	.439			.430					
		.447			.482					
		.533			.490					
		.432			.497					
		.486			.499					
		.560			.532					
CAUCHY	15	.303			.310					
		.327			.324					
		.404			.420					
		.276			.332					
		.365			.374					
		.389			.412					
EXPONENTIAL	15	.339			.329					
		.327			.392					
		.379			.385					
		.430			.449					
		.358			.374					
EMPIRICAL	15	16.660			16.940					
		17.040			18.860					
		19.200			19.980					
		19.860			21.630					
		17.090			16.740					

TABLE B16.

DESIGN (LANGLIE) AND ESTIMATION (NEXT STRESS)

RESPONSE CURVE	SAMPLE SIZE	ZERO NOISE SV & GW			NORMAL NOISE SV & GW			ASYMMETRIC NOISE SV & GW		
NORMAL	9,12,15	2.003 .423 1.088 .381 .960 1.101 .451 .759 .966	2.000 .344 .864 .311 .694 1.097 .442 .947 1.316	2.000 .361 1.255 .308 .775 .941 .405 .731 .903	2.001 .488 1.304 .378 .922 1.111 .445 .788 1.024	1.995 .383 .919 .351 .760 1.126 .462 .938 1.409	2.001 .370 1.332 .356 .801 .982 .432 .699 .904	1.951 .449 1.388 .406 .890 1.074 .448 .775 1.024	1.947 .421 .976 .379 .791 .372 .474 .898 1.418	1.945 .421 1.329 .381 .797 .958 .442 .725 .876
UNIFORM	15		1.297 .844 1.000 .476 .752 .914			1.305 .823 .972 .483 .811 .885				
CAUCHY	15		.497 .536 .764 .302 .574 .838			.476 .521 .712 .316 .556 .851				
EXPONENTIAL	15		.828 .381 .660 .856 1.300			.969 .399 .660 .931 1.239				
EMPIRICAL	15		48.890 18.620 30.240 38.380 53.800			52.440 19.370 29.560 39.530 46.810				

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